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Attractivity of solution with compact and non-compact semigroups for fractional evolution equation

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Abstract

We start the problem of atractivity of solutions for fractional evolution equation. We obtain some interesting results of mild solution for fractional evolution system with order $\beta \in (1, 2)$ in Banach space. There was a lot of circumstances for existence of universal attractive solution. We explain the Cauchy problems in these cases for which the semi-group is compact as well as non compact. Our results basically show some features of solution. We proceed the new representation of solution operators, by Laplace heat (is the new concept of light solution for objective equation), and Mainardi's Wright-type function then we go ahead to set up a new compact solution operators that contract results at the point when the sine family is compact.

Keywords: Fractional evolution equation, mild solution, Mainardi's Wright-type function attractivity, Riemann-Liuoville derivative, Caputo derivative.
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1 Introduction

Fractional differential equation has achieved fairly significance because of their applications in different sciences for example, Chemistry, Engineering, Mechanics, and Physics. In recent over the year, there has been a major break through, and partial differential equation involving fractional derivative. The existence theory of solutions for fractional evolution equation has been investigated intensively by numerous researchers like Kim [1], Podlubny [2], Kilbas [3], Zhou [4, 5, 6], Wang [7], Bazhlekova [8], Zacher [9], Zhou and Miller [10, 11]. They examine the attractivity of solution for Cauchy problems.

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There was a lot of circumstances for the existence of universal attractive solutions. It proved that all solutions are uniformly locally attracting. Lately, Chen [12], Losada [13], Tarasov [14] Banas and O'Regan [15] examine the attractivity of solutions for fractional ordinary differential equations. Anyway the finest thing of our insight there are relatively few results on the attractivity of solution for fractional evolution equation in liberal arts. Some authors S.Abbas, M. Benchohra [16], establish the result of attractivity of a coupled fractional Riemann-Liouville-Voltera-Stieltjies multi-delay partial integral system. Then authors shows that in coupled fraction the solutions are uniformly globally attractive. The existence of mild solution for integro-differential and fractional differential equation of order $\beta \in (1, 2)$ has attracted much attention in recent years.

Shu et al. [17] examine the existence of mild solution for non-local fractional differential calculation based on some sectorial operator. In this paper we discuss the existence and uniqueness of fractional theoretical Cauchy problem with order $\beta \in (1, 2)$. There are several ways to find fractional derivatives such as Raimann-Liouville, Caputo, Wayl, Hadmard, Grunwald-Letnikov.

Different authors have expressed their views on this topic in different ways. Due to which a revolution took place in the field of fractional differential equation. Continuing this process, we will work on the Caputo derivative and prove that the mild solution using initial value problems. Then we will create some assumptions which prove some important results. Consider Cauchy problem of fractional evolution equation with Caputo derivative:

$$\begin{cases} {}^{C}D_{0+}^{\beta} y(\nu) = By(\nu) + f(\nu, y(\nu)), & \nu \in [0, \infty), \\ y(0) = y_0, \quad y'(0) = y_1. & 1 < \beta < 2. \end{cases} \tag{1}$$

Where ${}^{C}D_{0+}^{\beta}$ is Caputo fractional derivative of order β, B is the infinitesimal generator of C_0-semigroup of bounded linear operator $\{R(\nu)\}_{\nu \geq 0}$ in Banach space Y and the time $\nu > 0$, $f : [0, \infty) \times Y \to Y$ is a particular function fulfills all assumptions, and y_0 is the component of the Banach space Y.

We establish some sufficient condition for the universal attractivity for mild solution in the study of semi-group is compact or non-compact. These results disclose the features of the solution for fractional evolution equation with Riemann-Liouville derivative. Since Mainardi's wright-type function is well-defined for $\beta \in (0, 1)$, how to define mild solution utilizing this function turns out to be more complicated and challenging. However by a cautious investigation we display another representation of solution operator by proposed work and another idea of mild solution is given. Then again we build up another new compact consequence of the solution operator at the point when the sine family is compact.

2 Preliminaries

In this part, we partially recall some concept of integration and derivative and then giving some theorems which are useful in the next section. Let Y be a Banach space with the $|\cdot|$. We indicate $L_b(Y,Z)$ are the interval of all bounded linear operators from Y to Z provided along with the norm $\|\cdot\|_{L_b(Y,Z)}$. Suppose that $L_b(Y) : Y \to Y$. Suppose that $C(H,Y)$ be the spaces of all continuous function from H to Y equipped with super norm $\|y\| = \sup_{\nu\in H} |y(\nu)|$. If $B : Y \to Y$ is a linear operator, we indicate the resolvent set of B by $\rho(B)$ and the resolvent of B by $Q(\lambda, B) = (\lambda I - B)^{-1} \in L_b(Y)$.

The fractional integral of order $\beta \in \mathbb{R}_+$ with zero lower limit for a function u is defined as

$$I^{\beta}_{0+}u(\nu) = g_\beta(\nu) * u(\nu) = \frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu - s)^{\beta-1}u(s)ds, \quad \nu > 0,$$

with the $*$ denote the convolution

$$g_\beta(\nu) = \frac{\nu^{\beta-1}}{\Gamma(\beta)},$$

where Γ is the regular gamma function. Just in case $\beta = 0$ we set $g_0(\nu) = \rho(\nu)$, the Dirac measure is converge at origin.

The RL-derivative of order $\beta \in \mathbb{R}_+$ with zero lower limit for a function $u : [0,\infty) \to \mathbb{R}$ is defined by

$${}^{L}D^{\beta}_{0+}u(\nu) = \frac{d^m}{d\nu^m}(g_{m-\beta} * s)(\nu),\ \nu > 0,\ m-1 < \beta < m,$$

and the similar Caputo's derivative of order $\beta \in \mathbb{R}_+$ with zero lower limit for a function $u : [0,\infty) \to \mathbb{R}$ is defined by

$${}^{C}D^{\beta}_{0+}u(\nu) = {}^{L}D^{\beta}_{0+}\left(u(\nu) - \sum_{k=0}^{m-1}\frac{u^{(k)}(0)}{k!}\nu^k\right), \quad \nu > 0, \quad m-1 < \beta < m.$$

The Wright function $M_\beta(\theta)$ is defined by

$$M_\beta(\theta) = \sum_{m=1}^{\infty}\frac{(-\theta)^{m-1}}{(m-1)!\Gamma(1-\beta m)},$$

It is realized that $M_\beta(\theta)$ satisfy the following equality:

$$\int_0^{\infty}\theta^{\delta}M_\beta(\theta)d\theta = \frac{\Gamma(1+\rho)}{\Gamma(1+\beta\rho)}, \quad \rho \geq 0,$$

we discuss some definition of mild solution

Definition 2.1. (*see*[18]): According to the mild solution of the Cauchy problem 1, we imply that the function $y \in C([0,\infty), Y)$ satisfies

$$y(\nu) = \nu^{\beta-1}P_\beta(\nu)y_0 + \int_0^\nu (\nu-s)^{\beta-1}P_\beta(\nu-s)f(s,y(s))ds, \quad \nu > 0,$$

where

$$P_\beta(\nu) = \int_0^\infty \beta M_\beta(\theta)Q(\nu^\beta\theta)d\theta.$$

Definition 2.2. : The mild solution $y(\nu)$ of the Cauchy problem 1 is attractive if $y(\nu)$ will in general to zero as $\nu \to \infty$.
Assume that B is the infinitesimal generator of a C_0-semigroup $\{R(\nu)\}_{\nu\geq 0}$ of equivalently bounded linear operators on Banach space Y. It implies that there exist $M \geq 1$ so that

$$M = \sup_{\nu\in[0,\infty)} \|R(\nu)\|_{W(Y)} < \infty,$$

where $W(Y)$ be the space of all bounded linear operators from Y to Y with the norm
$\|R(\nu)\|_{W(Y)} = sup\{|R(y)| : |y| = 1\}$, where $R \in W(Y)$ and $y \in Y$.

Proposition 2.1. (*see*[*18*]) : *For some fixed* $\nu > 0, P_\beta(\nu)$ *is bounded and linear operator, so that, for some* $y \in Y$,

$$|P_\beta(\nu)y| \leq \frac{M}{\Gamma(\beta)}|y|.$$

Proposition 2.2. (*see*[*18*]) : $\{P_\beta(\nu)\}_{\nu>0}$ *is strongly continuous, which implies that,* $\forall\ \ y \in Y$ *and* $\nu'' > \nu' > 0$, *we get*

$$|P_\beta(\nu'')y - P_\beta(\nu')y| \to 0, \quad \nu'' \to \nu'.$$

Proposition 2.3. (*see*[*18*]) : *Suppose that* $\{R(\nu)\}_{\nu>0}$ *is compact operator. At that point* $\{P_\beta(\nu)\}_{\nu>0}$ *is also compact operator. Suppose that* Y *be a real Banach space,* $I = [0,\infty)$:

$$D = \left\{ x \in C(H,Y) : \lim_{x\to\infty} \frac{|x(\nu)|}{1+\nu} = 0 \right\},$$

with the norm $\|x\| = \sup_{\nu\in[0,\infty)} \frac{|x(\nu)|}{(1+\nu)}$. *It is not difficult to see that* $(D, \|\cdot\|)$ *is a Banach space.*

Lemma 2.4. (*see*[[*19*], *Theorem* 1]) *The set* $I \subset C_0([0,\infty), Y)$ *is relatively compact as long as the following condition holds:*
(i) *For some* $c > 0$, *the function in* I *is equi-continuous on* $[0,c]$.
(ii) *For some* $\nu \in [0,\infty)$, $G(y) = \{y(\nu) : y \in G\}$ *is relatively compact in* Y.
(iii) $\lim_{t\to\infty} |y(\nu)| = 0$ *is uniformly for* $y \in I$.

$\mathbf{H_1}$: $|f(\nu,y)| \leq L\nu^{-\beta}$ for $y \in C((0,\infty), Y)$ and $\nu \in (0,\infty)$, where $\ 0 \leq L,\ \alpha k < \beta < 1$.

Lemma 2.5. (*see*[*20*]) : *consider* $0 < \beta k < \beta < 1$, *we can select* $\alpha > 0$ *sufficiently compact, thus*

$$\alpha - \beta + \beta k < 0 \quad and \quad \alpha + \beta k - \gamma < 0.$$

Let $T > 0$ *be sufficiently sizeable, thus*

$$M(y_0)T^{\alpha-\beta+\beta k} + M(y_1)T^{\alpha-\beta+\beta k} + \frac{ML\Gamma(\beta k)\Gamma(1-\beta)}{\Gamma(\beta k - \beta + 1)}T^{\alpha+\beta k-\gamma} \leq 1. \tag{2}$$

Define a set ω *given below*

$$\omega = \{y(\nu) : y \in C([0,\infty), Y),\ |y(\nu)| \leq \nu^{-\gamma},\ \nu \geq T\}.$$

It is not difficult to show that $\omega \neq \Phi$, *and* ω *is a bounded, closed, and convex subset of* $C_0((0,\infty), Y)$.

Definition 2.3. (*see*[21]) : we first define the Mittag-Leffler function $E_{\mu,\nu}(u)$ and Mainardi's Wright-type function $M_\varrho(u)$,

$$E_{\mu,\nu}(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\mu m + \nu)}, \quad 0 < \mu, \nu, \quad u \in \mathbb{C},$$

and

$$M_\varrho(u) = \sum_{n=0}^{\infty} \frac{(-u)^m}{m!\Gamma(1 - \varrho(m+1))}, \quad \varrho \in (0,1), \quad u \in \mathbb{C}.$$

Integrating step-by-step into Mittag-Leffler function

$$\int_0^\nu E_{\mu,\nu}(a\nu^\mu)\nu^{\nu-1}d\nu = \nu^\nu E_{\mu,\nu+1}(a\nu^\mu), \quad 0 < \mu, \nu, \quad a \in \mathbb{R}.$$

Lemma 2.6. (*see*[*21*]) : *For any fixed* $y \geq 0$, *and for some* $y \in Y$, *the subsequent evolution are valid*

$$|C_\beta(\nu)y| \leq M|y|, \quad |K_\beta(\nu)y| \leq M|y|\nu, \quad |P_\beta(\nu)y| \leq \frac{M}{\Gamma(2\beta)}|y|\nu^\beta.$$

Lemma 2.7. (*see*[*21*]) : *For some* $\nu > 0$,*the Mainardi's Wright-type function has the properties*

$$M_\varrho(\nu) \geq 0, \quad \int_0^\infty \theta^\delta M_\varrho(\theta)d\eta = \frac{\Gamma(1+\rho)}{\Gamma(1+\varrho\rho)}, \quad -1 < \rho < \infty,$$

and for $u \in \mathbb{C}$, $\mu \in (0,1)$

$$E_{\mu,1}(-u) = \int_0^\infty M_\mu(\theta)e^{-u\theta}d\theta, \quad E_{\mu,\mu}(-u) = \int_0^\infty \mu\theta M_\mu(\theta)e^{-u\theta}d\theta.$$

We assume that B *is an infinitesimal generator of a strongly continuous cosine family of equivalently bounded linear operator* $\{C(\nu)\}_{0<\nu}$ *in Banach space* Y, *there exist* $1 \leq M$ *so that*
$\|C(\nu)\|_{L_b(Y)} \leq M,\ 0 \geq \nu$. *For the purpose of simplification, we generally place* $\beta = \frac{\beta}{2}$ *for* $\beta \in (1,2)$. *We examine the linear nonhomogeneous fractional evolution system the system is identical for the*

following integral

$$\begin{cases} {}^C D^{\beta}_{0+} y(\nu) = By(\nu) + f(\nu, y(\nu)), & \nu \in [0, \infty), \\ y(0) = y_0, \quad y'(0) = y_1, & 1 < \beta < 2. \end{cases}$$

The above system is equal to the following integral

$$y(\nu) = y_0 + y_1\nu + \frac{1}{\Gamma(\beta)} \int_0^{\nu} (\nu - s)^{\beta-1}[By(s) + f(s)]ds, \quad \nu \in [0, \infty), \tag{3}$$

provided the integral 3 exist.

Theorem 2.8. *If 3 holds, then*

$$y(\nu) = C_\beta(\nu)y_0 + K_\beta(\nu)y_1 + \int_0^{\nu} (\nu - s)^{\beta-1} P_\beta(\nu - s)f(s)ds, \ \nu \in [0, \infty),$$

where

$$C_\beta(\nu) = \int_0^{\infty} M_\beta(\theta)C(\nu^\beta\theta)d\theta, \quad K_\beta(\nu) = \int_0^{\nu} C_\beta(s)ds, \quad P_\beta(\nu) = \int_0^{\infty} \beta\theta M_\beta(\theta)S(\nu^\beta\theta)d\theta$$

Proof. Let $\lambda > 0$

$$\xi(\lambda) \ = \ \int_0^{\infty} e^{-\lambda s}y(s)ds, \quad \mu(\lambda) = \int_0^{\infty} e^{-\lambda s} f(s, y(s))ds,$$

apply Laplace transformation on 3

$$\xi(\lambda) \ = \ \lambda^{\beta-1}(\lambda^\beta - A)^{-1}y_0 + \lambda^{\beta-2}(\lambda^\beta - A)^{-1}y_1 + (\lambda^\beta - A)^{-1}\mu(\lambda),$$

for $\nu \leq 0$

$$\xi(\lambda) \ = \ \lambda^{\frac{\beta}{2}-1} \int_0^{\infty} e^{-\lambda^{\frac{\beta}{2}}\nu}C(\nu)y_0 d\nu + \lambda^{-1}\lambda^{\frac{\beta}{2}-1} \int_0^{\infty} e^{-\lambda^{\frac{\beta}{2}}\nu}C(\nu)y_1 d\nu + \int_0^{\infty} e^{-\lambda^{\frac{\beta}{2}}\nu}S(\nu)\mu(\nu)d\nu.$$

Let

$$\Phi_\beta(\theta) = \frac{\beta}{\theta^{\beta+1}} M_\beta(\theta^{-\beta}), \quad \theta \in (0, \infty),$$

and its Laplace transform is given by

$$\int_0^{\infty} e^{-\lambda\theta}\Phi_\beta(\theta)d\theta = e^{-\lambda^\beta}, \quad \beta \in (\frac{1}{2}, 1)$$

$$\begin{aligned}
\lambda^{\beta-1}\int_0^\infty e^{-\lambda^\beta \nu} C(\nu) y_0 d\nu &= \int_0^\infty \beta(\lambda\nu)^{\beta-1} e^{-(\lambda\nu)^\beta} C(\nu^\beta) y_0 d\nu \\
&= \int_0^\infty -\frac{1}{\lambda}\frac{d}{d\nu}\Big(\int_0^\infty e^{-\lambda\nu\theta}\Phi_\beta(\theta)d\theta\Big) C(\nu^\beta) y_0 d\nu \\
&= \int_0^\infty \int_0^\infty \frac{-\lambda\theta}{-\lambda} e^{-\lambda\nu\theta}\Phi_\beta(\theta) C(\nu^\beta) y_0 d\nu \\
&= \int_0^\infty \int_0^\infty \theta\Phi_\beta(\theta) e^{-\lambda\nu\theta} C(\nu^\beta) y_0 d\nu d\theta \qquad (4)\\
&= \int_0^\infty \int_0^\infty \Phi_\beta(\theta) e^{-\lambda\nu} C(\frac{\nu^\beta}{\theta^\beta}) y_0 d\theta d\nu \\
&= \int_0^\infty e^{-\lambda\nu}\Big[\int_0^\infty \Phi_\beta(\theta) C(\frac{\nu^\beta}{\theta^\beta}) y_0\Big] d\theta d\nu \\
&= \mathcal{L}\Big[\int_0^\infty M_\beta(\theta) C(\nu^\beta\theta) y_0 d\theta\Big](\lambda) \\
&= \mathcal{L}[C_\beta(\nu) y_0](\lambda)
\end{aligned}$$

Since $L[g_1(\nu)](\lambda) = \lambda^{-1}$ according to Laplace Convolution Theorem, we get

$$\begin{aligned}
\lambda^{-1}\lambda^{\beta-1}\int_0^\infty e^{-\lambda^\beta \nu} C(\nu) y_1 d\nu &= \mathcal{L}[g_1(\nu)](\lambda) * \mathcal{L}[C_\beta(\nu) y_1](\lambda) \\
&= \mathcal{L}[(g_1 * C_\beta)(\nu) y_1](\lambda). \qquad (5)
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_0^\infty e^{-\lambda^\beta \nu} S(\nu)\mu(\lambda) d\nu &= \int_0^\infty \beta\nu^{\beta-1} e^{(-\lambda\nu)^\beta} S(\nu^\beta)\mu(\lambda) d\nu \\
&= \int_0^\infty \int_0^\infty \beta\nu^{\beta-1}\Phi_\beta(\theta) e^{-\lambda\nu\theta} S(\nu^\beta)\mu(\lambda) d\nu d\theta \\
&= \int_0^\infty \int_0^\infty \beta\frac{\nu^{\beta-1}}{\theta^\beta}\Phi_\beta(\theta) S(\frac{\nu^\beta}{\theta^\beta})\mu(\lambda) d\nu d\theta \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty e^{-\lambda^\beta \nu} S(\nu)\mu(\lambda) d\nu &= \int_0^\infty e^{-\lambda\nu}\Big[\int_0^\infty \beta\frac{\nu^{\beta-1}}{\theta^\beta}\Phi_\beta(\theta) S(\frac{\nu^\beta}{\theta^\beta})\mu(\lambda) d\theta\Big] d\nu \\
&= \mathcal{L}\Big[\int_0^\infty \beta\nu^{\beta-1} M_\beta(\theta) S(\nu^\beta\theta) d\theta\Big](\lambda).L[f(\nu)](\lambda) \\
&= \mathcal{L}\Big[\int_0^\nu (\nu-s)^{\beta-1} P_\beta(\nu-s) f(s) ds\Big](\lambda).
\end{aligned}$$

Combining equation 4, 5 and 6, we have

$$y(\nu) = C_\beta(\nu) y_0 + \int_0^\nu C_\beta(s) y_1 ds + \int_0^\nu (\nu-s)^{\beta-1} P_\beta(\nu-s) f(s, y(s)) ds.$$

Thus, the proof is complete. □

Lemma 2.9. *Assume that* (H_1) *hold. Then* $\{Zy : y \in \omega\}$ *is equi-continuous and* $lim_{x\to\infty}|(Zy)(\nu)| =$

0 *uniformly for* $y \in \omega$.

Proof. Consider $-\beta(1-k) < 0$ and $\beta k - \gamma < 0$ there exist $T_1 > T$ in such a way that
For any $y \in \omega$ and $T_1 < \nu_1, \nu_2$, we have

$$\begin{aligned} |(Zy)(\nu_2) - (Zy)(\nu_1)| &\leq \int_0^{\nu_2} (\nu_2 - s)^{\beta-1} |P_\beta(\nu_2 - s) f(s, y(s))| ds \\ &+ \int_0^{\nu_1} (\nu_1 - s)^{\beta-1} |P_\beta(\nu_1 - s) f(s, y(s))| ds, \end{aligned}$$

Furthermore for $0 < \nu_1 < \nu_2 \leq T_1$ Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} |(Zy)(\nu_2) - (Zy)(\nu_1)| &\leq \left| \int_0^{\nu_2} (\nu_2 - s)^{\beta-1} P_\beta(\nu_2 - s) f(s, y(s)) ds - \int_0^{\nu_1} (\nu_1 - s)^{\beta-1} P_\beta(\nu_1 - s) f(s, y(s)) \right| \\ &\leq \left| \int_0^{\nu_1} ((\nu_2 - s)^{\beta-1} - (\nu_1 - s)^{\beta-1}) P_\beta(\nu_2 - s) f(s, y(s)) ds \right| \\ &+ \left| \int_{\nu_1}^{\nu_2} (\nu_2 - s)^{\beta-1} P_\beta(\nu_2 - s) f(s, y(s)) ds \right| \\ &+ \left| \int_0^{\nu_1} (\nu_1 - s)^{\beta-1} \Big(P_\beta(\nu_2 - s) - P_\beta(\nu_1 - s)) f(s, y(s) \Big) ds \right| \\ &\leq M \sup_{\nu \in [0,T_1]} |f(\nu, y(\nu))| \int_0^{\nu_1} \Big[(\nu_1 - s)^{\beta k - 1} - (\nu_2 - s)^{\beta k - 1} \Big] ds \\ &+ M \sup_{\nu \in [0,T_1]} |f(\nu, y(\nu))| \int_{\nu_1}^{\nu_2} (\nu_2 - s)^{\beta k - 1} ds \\ &+ \int_0^{\nu_1} (\nu_1 - s)^{\beta-1} \left| \Big(P_\beta(\nu_2 - s) - P_\beta(\nu_1 - s) \Big) f(s, y(s)) \right| ds \\ &\leq \frac{M}{\beta k} \sup_{\nu \in [0,T_1]} |f(\nu, y(\nu))| [\nu_1^{\beta k} - \nu_2^{\beta k} + (\nu_2 - \nu_1)^{\beta k}] \\ &+ \frac{M}{\beta k} \sup_{\nu \in [0,T_1]} |f(\nu, y(\nu))| (\nu_2 - \nu_1)^{\beta k} \\ &+ \int_0^{\nu_1} (\nu_1 - s)^{\beta-1} \Big| P_\beta(\nu_2 - s) - P_\beta(\nu_1 - s) f(s, y(s)) \Big| ds, \end{aligned}$$

the above equation approaches to 0 as $\nu_2 \to \nu_1$.
If $0 = \nu_1 < \nu_2 \leq T_1$ we have $(Zy)(\nu_2) \to y_0 = (Fy)(0)$ as $\nu_2 \to 0$
Therefore collaborate the above contentions, it is obvious that the family of function $\{Zy : y \in \omega\}$ is equi-continuous. It remains to check that $lim_{\nu \to 0} |(Zy)(\nu)| = 0$ is uniformly for $y \in \omega$. We get

$$\begin{aligned} |(Zy)(\nu)| &\leq |C_\beta(\nu) y_0| + \int_0^\nu C_\beta(s) y_1 ds + \int_0^\nu (\nu - s)^{\beta k - 1} \big| P_\beta(\nu - s) f(s, y(s)) \big| ds \\ |(Zy)(\nu)| &\leq M|y_0| + M|y_1|\nu + \frac{ML\Gamma(\beta k)\Gamma(1-\beta)}{\Gamma(\beta k - \beta + 1)} \nu^{\beta k - \beta} \longrightarrow 0 \ as \ \nu \longrightarrow \infty, \end{aligned}$$

all the above discussion reveal that

$$\lim_{\nu\to\infty} |(Zy)(\nu)| = 0 \; is \; uniform \; for \; y \in s.$$

Thus, the theorem is proved. □

Lemma 2.10. *Suppose that (H_1) hold. At that point $\mathcal{Z}$ maps ω into ω and $\mathcal{Z}$ is continuous in ω.*

Proof. **Case I :**

$\mathcal{Z}$ maps ω into ω we know that $\mathcal{Z}y \in ([0,\infty), Y)$. Then again by utilizing condition H_1 we get

$$\begin{aligned}
|(\mathcal{Z}y)(\nu)| &\leq |C_\beta(\nu)y_0| + \int_0^\nu C_\beta(s)y_1 ds + \int_0^\nu (\nu-s)^{\beta-1}|P_\beta(\nu-s)f(s,y(s))|ds \\
&\leq \left(\nu^\alpha|C_\beta(\nu)y_0| + \nu^\alpha\int_0^\nu C_\beta(s)y_1 ds + \nu^\alpha\int_0^\nu (\nu-s)^{\beta-1}|P_\beta(\nu-s)f(s,y(s))|ds\right)\nu^{-\alpha} \\
&\leq \left(M|y_0|\nu^{\alpha-\beta+\beta k} + M|y_1|\nu^{\alpha-\beta+\beta k} + ML\nu^\alpha\int_0^\nu (\nu-s)^{\beta k-1}s^{-\gamma}ds\right)t^{-\alpha} \\
&\leq \left(M|y_0|\nu^{\alpha-\beta+\beta k} + M|y_1|\nu^{\alpha-\beta+\beta k} + \frac{ML\Gamma(\beta k)\Gamma(1-\beta)}{\Gamma(\beta k-\alpha+1)}\nu^{\alpha+\beta k-\gamma}\right)\nu^{-\alpha}
\end{aligned}$$

From the inequality 2, we get

$$\begin{aligned}
|(\mathcal{Z}y)(\nu)| &\leq \left(M|y_0|T^{\alpha-\beta+\beta k} + M|y_1|T^{\alpha-\beta+\beta k} + \frac{ML\Gamma(\beta k)(1-\beta)}{\Gamma(\beta k-\beta+1)}T^{\alpha+\beta k-\gamma}\right)\nu^{-\alpha} \\
&\leq \nu^{-\alpha}, \; \nu \geq T.
\end{aligned}$$

Which implies that $\mathcal{Z}\omega \subset \omega$.

Case II :

$\mathcal{Z}$ is continuous in ω for any $y_n, \; y \in \omega, \; n = 1, 2, ...$ with $lim_{x\to\infty} y_n = y$, we will indicate that $\mathcal{Z}y_n \to \mathcal{Z}_y$ as $n \to \infty$. For all $\epsilon > 0$ there exist $T_1 > T$.

In such a way that

$$\frac{ML\Gamma(\beta k)\Gamma(1-\beta)}{\Gamma(\beta k-\beta+1)}T_1^{\beta k-\beta} < \epsilon$$

At that point for $\nu > T_1$, we have

$$\begin{aligned}
|(\mathcal{Z}y_n)(\nu) - (\mathcal{Z}y)(\nu)| &\leq \int_0^\nu (\nu-s)^{\beta-1}\left|P_\beta(\nu-s)\Big(f(s,y_n(s)) - f(s,y(s))\Big)\right|ds \\
&\leq M\int_0^\nu (\nu-s)^{\beta k-1}\Big(|f(s,y_n(s))| + |f(s,y(s))|\Big)ds \\
&\leq ML\int_0^\nu (\nu-s)^{\beta k-1}s^{-\gamma}ds \\
&\leq \frac{ML\Gamma(\beta k)\Gamma(1-\beta)}{\Gamma(\beta k-\beta+1)}T_1^{\beta k-\beta} < \epsilon,
\end{aligned}$$

for $0 < \nu \leq T_1$ we get

$$\begin{aligned} |(\mathcal{Z}y_n)(\nu) - (\mathcal{Z}y)(\nu)| &\leq \int_0^\nu (\nu - s)^{\beta-1} \Big| P_\beta(\nu - s)\Big(f(s, y_n(s)) - f(s, y(s))\Big)\Big| ds \\ &\leq M \int_0^\nu (\nu - s)^{\beta k-1}\Big(|f(s, y_n(s))| - |f(s, y(s))|\Big) ds. \end{aligned}$$

Seeing that $\lim_{n\to\infty} |f(\nu, y_n(\nu)) - f(\nu, y(\nu))| = 0$, by Lebesgue dominated convergence theorem we get

$$|(\mathcal{Z}y_n)(\nu) - (\mathcal{Z}y)(\nu)| \longrightarrow 0, \ n \to \infty.$$

Thus, clearly

$$\|(\mathcal{Z}y_n)(\nu) - (\mathcal{Z}y)(\nu)\| \longrightarrow 0, \ n \to \infty.$$

Which infers that the operator $\mathcal{Z}$ is continuous.

Thus the proof is complete. □

3 Compact Semigroup Case

We assume that the operator $S(\nu)$ is compact for $0 < \nu$.

Theorem 3.1. *Suppose that $(H1)$ hold. At that point the Cauchy problem 1 concedes at least one attractive solution.*

Proof. Clearly y be a mild solution of 1 in ω as long as y be a fixed point of Z in ω. Therefore, it is sufficient to prove that the operator Z has a fixed point in ω. According to lemma 2.10 it states that $Z : \omega \to \omega$ is continuous and bounded. Then, it must be prove that Z is relatively compact. One can surmise from lemma 2.9 that $\{Zy : y \in \omega\}$ is equi-continuous as well as $lim_{x\to\infty}|(Zy)(\nu)| = 0$ uniformly for $y \in \omega$. It remain to prove that $W(t) = \{(Zy)(\nu) : y \in \omega\}$ is relatively compact in Y. Suppose that $t \in [0, \infty)$.

Clearly, $W(0)$ is relatively compact in Y. Suppose that $\nu \in (0, \infty)$ be fixed, for all $\varrho > 0$ and for all $\zeta > 0$, describe an operator $Z_{\varrho,\zeta}$ on ω as given below:

$$\begin{aligned} (Z_{\epsilon,\zeta}y)(\nu) &= C_\beta(\nu)y_0 + K_\beta(\nu)y_1\nu + \int_0^{\nu-\varrho}\int_\zeta^\infty \beta\theta(\nu - s)^{\beta-1}M_\beta(\theta) \times S((\nu - s)^\beta\theta)f(s, y(s))d\theta ds \\ &= C_\beta(\nu)y_0 + K_\beta(\nu)y_1\nu + S(\varrho^\beta\zeta)\int_0^{\nu-\varrho}\int_\zeta^\infty \beta\theta(\nu - s)^{\beta-1}M_\beta(\theta) \\ &\quad \times S((\nu - s)^\beta\theta - \varrho^\beta\theta)f(s, y(s))d\theta ds. \end{aligned}$$

Then, at that point through the compactness of $C_\beta(\nu)$, $K_\beta(\nu)$ and $S(\varrho^\beta\zeta)(\varrho^\beta\zeta > 0)$, we get $W_{\varrho,\zeta}(\nu) = \{(Z_{\varrho,\zeta}y) : y \in \omega\}$ is relatively compact in Y $\forall$ $\varrho \in (0, \nu)$ and $\forall$ $\zeta > 0$. Furthermore, for each $y \in \omega$, we get

$$\begin{aligned}
|(Zy)(\nu) - (Z_{\varrho,\zeta}y)(\nu)| &\leq \left| \int_0^{\nu} \int_0^{\zeta} \beta\theta(\nu - s)^{\beta-1} M_\beta(\theta) S((\nu - s)^\beta \theta) f(s, y(s)) d\theta ds \right| \\
&+ \left| \int_{\nu-\varrho}^{\nu} \int_{\zeta}^{\infty} \beta\theta(\nu - s)^{\beta-1} M_\beta(\theta) S((\nu - s)^\beta \theta) f(s, y(s)) d\theta ds \right| \\
&\leq \beta M_0 \int_0^{\nu} (\nu - s)^{\beta k-1} |f(s, y(s))| ds \int_0^{\zeta} \theta M_\beta(\theta) d\theta \\
&+ \beta M_0 \int_{\nu-\varrho}^{\nu} (\nu - s)^{\beta k-1} |f(s, y(s))| ds \int_0^{\infty} \theta M_\beta(\theta) d\theta \\
&\leq \beta M_0 L \int_0^{\nu} (\nu - s)^{\beta k-1} s^{-\gamma} ds \int_0^{\zeta} \theta M_\beta(\theta) d\theta \\
&+ \beta M_0 L \int_{\nu-\varrho}^{\nu} (\nu - s)^{\beta k-1} s^{-\gamma} ds \int_0^{\infty} \theta M_\beta(\theta) d\theta \\
&\leq \frac{\beta M_0 L \Gamma(\beta k) \Gamma(1 - \beta)}{\Gamma(\beta k - \beta + 1)} \nu^{\beta k-\gamma} \int_0^{\zeta} \theta M_\beta(\theta) d\theta \\
&+ \beta M_0 L \nu^{\beta k-\gamma} \int_{\frac{1-\varrho}{\nu}}^{1} (1 - s)^{\beta-1} s^{-\gamma} ds \int_0^{\infty} \theta M_\beta(\theta) d\theta \to 0 \ as \ \varrho \to 0, \ \zeta \to 0.
\end{aligned}$$

Hence, there exists relatively compact sets arbitrary close to set $W(\nu)$. Therefore, $W(\nu)$ is also relatively compact in Y. At last according to Schauder fixed point theorem, 1 has a mild solution $y \in \omega$ as well as $y(\nu) \to 0$ *as like* $\nu \to \infty$.

Thus, the proof is complete. □

4 Noncompact Semigroup Case

When $S(\nu)$ is non-compact, we define the suppositions given below:

(H2): $f : [0, T] \times Y \to Y$ is a Caratheodory function and for some $p > 0$ there exist a suitable function $m_p(\nu) \in L^q((0, T), R^+)$ with $q > -\frac{1}{\beta\gamma}$ in such a way that

$$|f(\nu, y)| \leq m_p(\nu), \quad and \quad \lim_{p \to +\infty} \inf \frac{|m_p(\nu)|_{L^q(0,T)}}{p} = \eta < \infty,$$

for $\nu \in [0, T]$ and $\forall\ y \in Y$ satisfy $|y| \leq p$.

At that point, for each $y_0, y_1 \in B(A^\alpha)$ with $\alpha < 1 + \gamma$, the problem 1 has at least one mild solution, provide that

$$C_q \eta \left(\frac{T^{1-(1+\beta\gamma)u}}{1 - (1 + \beta\gamma)u} \right)^{\frac{1}{u}} < 1,$$

where $u = \frac{q}{q-1}$.

Theorem 4.1. *Suppose that* $(H1)$ *and* $(H2)$ *holds. Then, at that point the Cauchy problem 1 have* *at least one attractive solution.*

Proof. According to lemma 2.10 we see that $Z : \omega \to \omega$ is continuous as well as bounded. Then, it will be prove that Z is relatively compact. One can surmise from lemma 2.9 that $\{Zy : y \in \omega\}$ is equi-continuous as well as $lim_{x\to\infty}|(Zy)(\nu)| = 0$ is uniformly for $y \in \omega$. It remains to prove that $W(\nu) = \{(Zy)(\nu) : y \in \omega\}$ is relatively compact in Y.
For $\nu \in [0, T]$, $\{(Zy)(\nu) : y \in \omega\}$ is pre-compact in Y. When $\nu = 0$ it is enough to see that $\{(Zy)(0) : y \in \omega\} = \{y_0, y_1 : y \in \omega\}$ is compact. Suppose that $\nu \in [0, T]$ be fixed as well as $\varrho, \zeta > 0$. For $v \in \omega$, we define an operator $Z_{\varrho,\zeta}$ by

$$(Z_{\varrho,\zeta}y)(\nu) = C_\beta(\nu)y_0 + K_\beta(\nu)y_1\nu + \int_0^{\nu-\varrho}\int_\zeta^\infty \beta\theta(\nu - s)^{\beta-1}M_\beta(\theta)T((\nu - s)^\beta\theta)f(s, y(s))d\theta ds,$$

A has compact resolvent, for each $\nu \in (0, T]$, $\{(Z_{\varrho,\zeta}y)(\nu) : y \in \omega,\ \varrho > 0,\ 0 < \zeta < \nu\}$ is pre-compact in Y. Using (H2)

$$\begin{aligned}
|(Zy)(\nu) - (Z_{\varrho,\zeta}y)(\nu)| &\leq \left|\int_0^\nu\int_0^\zeta \beta\theta(\nu - s)^{\beta-1}M_\beta(\theta)T((\nu - s)^\beta\theta)f(s, y(s))d\theta ds\right| \\
&+ \left|\int_{\nu-\varrho}^\nu\int_\zeta^\infty \beta\theta(\nu - s)^{\beta-1}M_\beta(\theta)T((\nu - s)^\beta\theta)f(s, y(s))d\theta ds\right| \\
&\leq \int_0^\nu C_q(\nu - s)^{-1-\beta\gamma}m_p(s)ds\int_0^\zeta \theta^{-\gamma}M_\beta(\theta)d\theta \\
&+ \int_{\nu-\varrho}^\nu C_q(\nu - s)^{-1-\beta\gamma}m_p(s)ds\int_\zeta^\infty \theta^{-\gamma}M_\beta(\theta)d\theta \\
&\leq C_q\left(\frac{T^{1-(1+\beta\gamma)u}}{1 - (1 + \beta\gamma)u}\right)^{\frac{1}{u}}\|m_p\|_{L^q(0,T)}\int_0^\zeta \theta^{-\gamma}M_\beta(\theta)d\theta \\
&+ C_q\left(\frac{T^{1-(1+\beta\gamma)u}}{1 - (1 + \beta\gamma)u}\right)^{\frac{1}{u}}\|m_p\|_{L^q(0,T)}\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma\beta)}
\end{aligned}$$

Utilize the total boundedness we have for every $\nu \in (0, T]$, $\{(Zy)(\nu) : y \in \omega\}$ is pre-compact in Y. Hence, for every $\nu \in (0, T]$, $\{(Zy)(\nu) : y \in \omega\}$ is pre-compact in Y. Finally, Schauder fixed point theorem, 1 has a mild solution $y \in \omega$ as well as $y(\nu) \to 0$, $\nu \to \infty$.
Thus, the proof is complete.

□

5 Example

Suppose that $\omega \subset \mathbb{R}^N$ be a bounded domain $(R \geq 0)$ with boundary $\partial\omega$ of group G^4.
Examine the fractional initial bounded value problem

$$\begin{cases} ({}^C D^{\beta}_{0+} y)(\nu, u) = \delta y(\nu, u) + f(\nu, y(\nu, u)), \quad \nu > 0,\ u \in \omega, \\ \frac{v}{\partial \omega} = 0, \\ y(0, u) = y_0(u), \quad y'(0, u) = y_1(u). \end{cases} \tag{7}$$

In space $G^b(\overline{\omega})$ $(1 < l < 2)$, where δ represent the Laplacian operator as regards the spatail variable as well as ${}^C D^{\beta}_{0+}$ acts for the Caputo fractional derivative of order $\beta(1 < \beta < 2)$. Place

$$\begin{aligned} \tilde{B} &= \delta \\ D(\tilde{B}) &= \{y \in G^{2+b}(\overline{\omega}) : y = 0 \ \ on \ \ \partial\omega\}. \end{aligned}$$

It follows from that $[[22], Example\ 2.3]$ the point exist $\mu, \epsilon > 0$, such that

$$\tilde{B} + \mu \in \Theta^{\frac{l}{2}-1}_{\frac{\Pi}{2}-\epsilon}\Big(G^b(\overline{\omega})\Big).$$

Then problem 7 can be composed dynamically as $(see$ $[[23], Example 6.2])$

$$\begin{cases} {}^C D^{\beta}_{0+} y(\nu) = By(\nu) + f(\nu, y(\nu)), & \nu \in [0, \infty) \\ y(0) = y_0, \quad y'(0) = y_1, & 1 < \beta < 2. \end{cases}$$

Suppose that $f(\nu, y(\nu)) = \nu^{-\beta} \sin y(\nu)$. At that point the supposition $(\mathbf{H_1})$ that is obviously fulfilled. In outcomes, as indicated by lemma 2.9, the problem 7 has at any rate one attractive mild solution.

Then again, for the first order evolution equation

$$\begin{cases} y'(\nu) = By(\nu) + \nu^{-\beta}, & \nu \in [0, \infty) \\ y(0) = y_0, \quad y'(0) = y_1, & 1 < \beta < 2. \end{cases}$$

Our result basically disclose certain attributes of solutions for fractional evolution equation, which are not controlled by integer order evolution equations.

6 Conclusion

The principal finding of this work show the specific class of attractivity solutions for fractional evolution equation, we establish sufficient conditions for global attractivity of mild solution, while the integer order evolution equation don't have such attractivity. We discuss the attractivity of solutions for Cauchy problems. Cauchy problems in these cases for which the semi-group is compact as well as non compact. However there are not many attractive solution for fractional evolution equation in literature.

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References

[1] I. Kim, K. H. Kim, S. Lim, $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients. Advances in Mathematics, 306, 123–176, (2017).

[2] I. Podlubny, Fractional differential equations, Academic Press, San Diego, (1999).

[3] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations, Elsevier, 204, (2006).

[4] Y. Zhou, Fractional evolution equations and inclusions: Analysis and Control, Academic Press, Elsevier, (2016).

[5] Y. Zhou, L. Peng, On the time-fractional Navier-Stokes equations. Computers & Mathematics with Applications, no. 6, 73, 874âĂŞ891, (2017).

[6] Y. Zhou, L. Zhang, Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems. Computers & Mathematics with Applications, no. 6, 73, 1325âĂŞ1345, (2017).

[7] J. Wang, Y. Zhou, W. Wei, A class of fractional delay nonlinear integro-differential controlled systems in Banach spaces. Communications in Nonlinear Science & Numerical Simulation, no. 10, 16, 4049–4059, (2011).

[8] E. Bazhlekova, The abstract Cauchy problem for the fractional evolution equation. Fractional Calculus & Applied Analysis, no. 3, 1, 255âĂŞ270, (1998).

[9] R. Zacher, A De Giorgi-Nash type theorem for time fractional diffusion equations. Mathematics Analysis, no. 1, 356, 99âĂŞ146, (2013).

[10] Y. Zhou, L. Zhang, and X. H. Shen, Existence of mild solutions for fractional evolution equations. Integral Equations & Applications, no. 4, 25, 557âĂŞ586, (2013).

[11] K. S. Miller, B. Ross, An introduction to the fractional calculus and differential equations. John Wiley, New York, 1993.

[12] F. Chen, J. J. Nieto, and Y. Zhou, Global attractivity for nonlinear fractional differential equations. Nonlinear Analysis: Real World Applications, no. 1, 13, 287âĂŞ298, (2012).

[13] J. Losada, J. J. Nieto, and E. Pourhadi, On the attractivity of solutions for a class of multi-term fractional functional differential equations. Computational & Applied Mathematics, 312, 2âĂŞ12, (2017).

[14] V. E. Tarasov, Fractional Dynamics: Application of fractional calculus to dynamics of particles. Fields and Media, Springer, (2010).

[15] J. BanaÂťs and D. OâĂŹRegan, On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order. Mathematical Analysis & Applications, no. 1, 345, 573âĂŞ582, (2008).

[16] S.Abbas, N. Hamaidi, M. Benchohra, G. N Gurrekata, Existence and attractivity results for coupled systems of nonlinear VolterraâĂŞStieltjes Multi-delay fractional partial integral equations. Abstract & Applied Analysis, 2018, Article ID 8735614, 10, (2018).

[17] X. B. Shu, Q. Wang, The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Computers & Mathematics with Applications, no. 6, 64, 2100-2110, (2012).

[18] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, Singapore, (2014).

[19] W. Guo, A generalization and application of Ascoli-Arzela theorem, Systems Science & Mathematics Science, no. 1, 22, 115–122, (2002).

[20] Y. Zhou, Attractivity for fractional evolution equations with almost sectorial operators. Fractional Calculus & Applied Analysis, no. 3, 21, 786–800, (2018).

[21] Y. Zhou and J. Wei, New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$. Evolution Equations & Control Theory, no. 3, 10, 491, (2021).

[22] F. Mainardi, P. Paraddisi, R. Gorenflo, Probability distribution generated by fractional diffusion equation. In: Economics: An Emerging Science, J. Kertesz, I. Kondor (Eds.), Kluwer, Dordrecht (2000).

[23] R. N. Wang, D. H. Chen, T.J. Xiao, Abstract fractional Cauchy problems with almost sectorial operator. Differential Equations, no. 1, 252, 202-235, (2012).