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Non-Compact Semigroups and Controllability of Fractional Evolution Equations of Order (1, 2)

Syed Zargham Haider Sherazi[1], **Hafiza Maria Arshad**[2], **Mehwish Iqbal**[1], **Muhammad Usman Mehmood**[1†]

Abstract This work present the controllability of fractional evolution equations of order (1, 2). We use the fractional calculus, the Mönch fixed point (MFP) theorem and measure of non-compactness (MNC). A controllability result is given out for the nonlocal Cauchy problem of the fractional evolution equations including noncompact semigroups (NCSG) and the functions by excluding Lipschitz continuity. The associated theorems and properties are demonstrated in detail and an example is stated to clarify the effectiveness of the theoretical outcomes.

Keywords: Fractional evolution equation, Controllability, MNC, MFP theorem, Nonlocal Cauchy (NLC) problem
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1. Introduction

Controllability is the basic theory in control theory of mathematical, which assumes a significant part in control systems. In the last few years, different methods have been used in many publications to study the controllability of distinct nonlinear

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[†]Corresponding author: usmanwinning89@gmail.com
[1] Department of Mathematics and Statistics, The University of Lahore, Sargodha, Pakistan.
[2] OPF Bhalwal, Sargodha, Pakistan

and linear stochastic and deterministic dynamic systems. (cf., e.g., (1; 2; 3; 4; 5; 6; 7; 8; 9; 10)). There have additionally been a few papers (e.g, (11; 12; 13)) in which non-linear evolution equations described the controllability of systems in infinite dimensional spaces. Further when we deal with compact semigroup and other assumptions are satisfied, the utilization of controllability outcomes is simply confined to the finite dimensional space (cf. (14; 15; 16)).

In this article, we present the controllability of fractional evolution equations of order (1, 2). We use MNC, the fractional calculus and the MFP theorem. We get a controllability results of the fractional evolution equations for the nonlocal Cauchy problem including NCSG and the nonlocal functions by excluding Lipschitz continuity. Let us assume the nonlinear fractional evolution system:

$$\begin{cases} {}^{c}\mathscr{D}^{a}u(\zeta)+Au(\zeta)=f(\zeta,u(\zeta),Gu(\zeta))+Bv(\zeta),\ \zeta\in I \\ u(0)=A^{-1}(0)g(u) \\ u'(0)=u_{1}. \end{cases} \tag{1}$$

where ${}^{c}\mathscr{D}^{a}$ is the Caputo fractional derivative of order $a\in(1,2), -A:\mathscr{D}(A)\subset U\rightarrow U$ is the infinitesimal generator of a C_0-semigroup $\delta(\zeta)(\zeta\geqslant 0)$ of uniformly bounded linear operator B, u is the control function given in $L^2(I,U)$ operator $I=[0,a]$, $a>0$ is a constant, define the family of closed linear operators $A(\zeta)$ on a dence domain $\mathscr{D}(A)$ in Banach space E' into E', where $D(A)$ is independent of ζ, The continuous function $f:I\times E'\rightarrow E'$, $g:C(I,E')\rightarrow E'$ is a non-local function to be specified and
$Gu(\zeta)=\int_0^{\zeta}K(\zeta,\varpi)u(\varpi)d\varpi$.
is a Voltera integral operator kernel $K\in C(\varpi,\mathbb{R}^{+}),\varpi=\{(\zeta,\varpi):e\geqslant\zeta\geqslant\varpi\geqslant 0\}$. All through this article we generally expect that
$K^{*}=\sup_{\zeta\in I}\int_0^{\zeta}K(\zeta,\varpi)d\varpi$.
In this paper, we use a major type of nonlocal function . At the start, we familiarize a reasonable definition of mild solutions of the system (1), and then in noncompact semigroup the controllability of the system (1) is determined by applying the MFP theorem. The remainig paper is precised as follows. In Section 2, as regard to fractional calculus and MNC some preliminaries are given. In Section 3, we provide the controllability results of the fractional evolution equation system (1). Toward the end,

an applicatin is presented to demonstrate the theory of the obtained outcomes.

2. Preliminaries

Let U is a Banach space with norm $\|.\|$. We represent the Banach space by $C(I,U)$ $\forall$ continuous U-value function on interval I with the norm $\|u\|_c = max\|u(\zeta)\|$. We take $\|f\|_{L^{\wp}}$ to represent the $L^{\wp}(I,\mathbb{R}^+)$ norm of $f \in L^{\wp}(I,\mathbb{R}^+)$ for some $\wp$ with $1 \leqslant \wp < \infty$. Let a closed linear operator $A : \mathscr{D}(A) \subset U \to U$ and $-A$ is a generator of C_0-semigroup $\triangle(\zeta)(\zeta \geqslant 0)$ of uniformly bounded linear operator in U. Therefore, $\exists$ a constant $N > 1$ such that $\|\wp(\zeta)\| \leqslant N$ for all $\zeta \geqslant 0$.

Definition 1. Taking the lower limit as zero for a function $f \in C^m[o,\infty)$ the Caputo fractional derivative of order $n-1 < \alpha < n$ can be written as

$$^c\mathscr{D}^a f(\zeta) = \frac{1}{\Gamma(m-a)} \int_0^{\zeta} (\zeta - \varpi)^{m-a-1} f^{(m)}(\varpi)d\varpi, t > 0, m \in \mathbb{N}.$$

For $x \in U$, we define three families

$\{C_a(\zeta)\}_{\zeta \geqslant 0}$, $J_a(\zeta) = \int_0^{\zeta} C_a(\zeta)d\zeta$ and $\{K_a(\zeta)\}_{0 \leqslant \zeta}$ of operators by

$C_a = \int_0^{\infty} M_a(\theta)c(\zeta^a\theta)d\theta$, $\quad K_a(\zeta) = \int_0^{\infty} a\theta M_a\theta\varpi(\zeta^a\theta)d\theta$, $\quad 0 < a < 1$

where

$M_a(\theta) = \dfrac{1}{a}\theta^{-1-\frac{1}{a}}\rho_a(\theta^{-\frac{1}{a}})$,

$\rho_a(\theta) = \dfrac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\theta^{-am-1}\dfrac{\Gamma(ma+1)}{m!}\sin(m\pi a)$, $\quad (0,\infty) \in \theta$. The probability density function M_a defined on $(0,\infty)$ which has the following properties $M_a(\theta) \geqslant 0$ for all $(0,\infty) \in \theta$ and $\int_0^{\infty} M_a(a)d\theta = 1$.

Lemma 1. The operators $C_a(\zeta), J_a(\zeta)$ and $K_a(\zeta)$ proceed the listed characteristic.

(i) For each fixed $0\zeta \geqslant 0$ and any $x \in U$.
$\|C_a(\zeta)x\| \leqslant N\|x\|$, $\quad \|K_a(\zeta)x\| \leqslant \dfrac{N}{\Gamma(a)}\|x\|\zeta^a$, $\quad \|J_a(\zeta)x\| \leqslant N\|x\|\zeta$.

(ii) For all $0 \leqslant \zeta$ the operators $C_a(\zeta)$,$K_a(\zeta)$ and $J_a(\zeta)$ are strongly continuous.

(iii) If we have an equicontinuous semigroup $\{\triangle(\zeta)(0 \leqslant \zeta)\}$, then $C_a(\zeta)$,$K_a(\zeta)$ and $J_a(\zeta)$ in U for 0<t.

Definition 2. (see, (17), (18), (19)). Let in Banach space $(E', \leqslant)$ take a non-negative cone Θ^+ of an order (1,2). Defined a function Υ on the set of all bounded subsets of the Banach space U with values in Θ^+ is called measure of non-compactness (MNC) on U if $\varphi(\overline{c_0}(\Psi)) = \Upsilon(\Psi)$ for all bounded subset $\Psi \subset U$, where $\overline{c_0}(\Psi)$ is called closed convex hull of Ψ.

The major case of the measure of non-compactness of Hausdorff $\mathfrak{B}$ defined by $\mathfrak{B}(\Psi) = \inf\{0 < \varepsilon : \varepsilon$ is greater than a finite number of balls of radii which cover the $\Psi\}$ on each bounded subset Ψ of U. For any $C(I,U) \subset B$ and $\zeta \in I$, set $B(\zeta) = \{u(\zeta) : u \in B\} \subset U$. If in $C(I,U)$ B is bounded, then also in U $B(\zeta)$ is bounded and $\mathfrak{B}(B) \geqslant \mathfrak{B}(B(\zeta))$. It is notable that the measureof noncompactness $\mathfrak{B}$ verifies the listed characteristics, (see, e.g, (20), (21), (22)) for all bounded subsets Ψ, Ψ_1, Ψ_2 of U.

(1) $\Psi_1 \subset \Psi_2 \Rightarrow \mathfrak{B}(\Psi_2) \geqslant \mathfrak{B}(\Psi_1)$;

(2) $\mathfrak{B}(\Psi_1) + (\Psi_2) \geqslant \mathfrak{B}(\Psi_1 + \mathfrak{B}\Psi_2)$ *where* $\Psi_1 + \Psi_2 = \{x + y : x \in \Psi_1, y \in \Psi_2\}$;

(3) $\max\{\beta(\Psi_1), (\Psi_2)\} \geqslant \beta(\Psi_1 + \Psi_2)$;

(4) $|\lambda|\mathfrak{B}(\Psi) \geqslant \mathfrak{B}(\lambda\Psi)$ *for any* $\lambda \in \mathbb{R}$.

(5) $\mathfrak{B}(\{a\} \cup \Psi) = \mathfrak{B}(\Psi)$ *for every* $a \in U$.

(6) $\mathfrak{B}(\Psi) = 0 \Longleftrightarrow \Psi$ is relatively compact in U.

Definition 3. $x \in C(I,U)$ is a function which defined a mild solution of system (1) if for each $u \in L^2(I,U)$ the integral equation

$$\begin{aligned} u(\zeta) &= C_a(t)A^{-1}(0)g(u) + J_a(\zeta)u_1 + \int_0^\zeta (\zeta - \varpi)^{a-1} K_a(\zeta - \varpi)\Big[f\big(\varpi, u(\varpi), Gu(\varpi)\big) \\ &\quad + Bv(\varpi)\Big] d\varpi. \end{aligned}$$

is satisfied.

3. Main results

Theorem 1. We get a controllability result of the fractional evolution equations for the NLC problem including NCSG and the nonlocal functions by excluding Lipschitz continuity. Let us assume the nonlinear fractional evolution system:

$$\begin{cases} {}^{c}\mathscr{D}^{a}u(\zeta)+Au(\zeta)=f(\zeta,u(\zeta),Gu(\zeta))+Bv(\zeta), \ \ \zeta\in I \\ u(0)=A^{-1}(0)g(u) \\ u'(0)=u_1. \end{cases}$$

where ${}^{c}\mathscr{D}^{a}$ is the Caputo fractional derivative of order $1<a<2$, operator I=$[0,a]$, $a>0$ is a constant.

Proof. We can write the integral equation

$$u(\zeta)=A^{-1}(0)g(u)+u_1t+\frac{1}{\Gamma(a)}\int_0^{\zeta}(\zeta-\varpi)^{a-1}[f(\varpi,u(\varpi),Gu(\varpi))+Bv(\varpi)-Au(\varpi)]d\varpi. \tag{2}$$

$$u(\zeta) = C_a(t)A^{-1}(0)g(u)+J_a(\zeta)u_1+\int_0^{\zeta}(\zeta-\varpi)^{a-1}k_a(\zeta-\varpi) \tag{3}$$

$$\left[f\left(\varpi,u(\varpi),Gu(\varpi)+Bv(\varpi)-Au(\varpi)\right)\right]d\varpi. \tag{4}$$

where $C_a(\zeta)=\int_0^{\infty}M_a(\theta)c(\zeta^a\theta)d\theta$,
$I_a(\zeta)=\int_0^{t}C_a(\varpi)d\varpi$,
$K_a(\zeta)=\int_0^{\infty}a\theta M_a\theta\varpi(\zeta^a\theta)d\theta$.
Let $\lambda>0$ and the Laplace Transform is given

$$\begin{aligned} \nu(\lambda)=L[u(\zeta)](\lambda) &= \int_0^{\infty}e^{-\lambda\varpi}u(\varpi)d\varpi \\ \text{and} \qquad \mu(\lambda) &= L\left[f\left((\zeta),u(\zeta),Gu(\zeta)\right)+Bv(\zeta)-Au(\zeta)\right](\lambda) \\ &= \int_0^{\infty}e^{-\lambda\varpi}\left[f\left((\zeta),u(\zeta),Gu(\zeta)\right)+Bv(v)-Au(v)\right]d\varpi. \end{aligned}$$

Now taking Laplace Transform of (2)

$$\begin{aligned} L[u(\zeta)] &= L[A^{-1}(0)g(u)] + L[u_1\zeta] + L\Bigg[\frac{1}{\Gamma(a)}\int_0^{\zeta}(\zeta-\varpi)^{a-1} \\ &\quad \big[f\big((\varpi),u(\varpi),Gu(\varpi)\big) + Bv(\varpi) - Au(\varpi)b\big]d\varpi\Bigg] \\ v(\lambda) &= \frac{1}{\lambda}A^{-1}(0)g(u) + \frac{1}{\lambda^2}u_1 - \frac{1}{\lambda^a}Av(\lambda) + \frac{1}{\lambda^a}\mu(\lambda)v(\lambda) + \frac{1}{\lambda^a}Av(\lambda) \\ &= \frac{1}{\lambda}A^{-1}(0)g(u) + \frac{1}{\lambda^2}u_1 + \frac{1}{\lambda^a}\mu(\lambda)v(\lambda)\left[\frac{\lambda^a I + A}{\lambda^a}\right] \\ &= \frac{1}{\lambda}A^{-1}(0)g(u) + \frac{1}{\lambda^2}u_1 + \frac{1}{\lambda^a}\mu(\lambda) \\ v(\lambda) &= (\lambda^a I + A)^{-1}\lambda^{a-1}A^{-1}(0)g(u) + (\lambda^a I + A)^{-1}\lambda^{a-2}u_1 \\ &\quad + (\lambda^a I + A)^{-1}\mu(\lambda). \end{aligned} \tag{5}$$

Note: For any $\lambda > 0$ there exist a bounded inverse operator $[\lambda^{\mathfrak{B}} I + A]^{-1} \in L[E]$ and

$$||[\lambda^{\beta} I + A]^{-1}|| < \frac{C}{|\lambda| + 1}, \; If \; \zeta > 0$$

.

$$\begin{aligned} v(\lambda) &= \lambda^{\frac{\mathfrak{B}}{2}-1}\int_0^{\infty} e^{-\lambda^{\frac{\mathfrak{B}}{2}}\zeta} c(\zeta)\big[A^{-1}(0)g(u)\big]d\zeta + \lambda^{-1}\lambda^{\frac{\mathfrak{B}}{2}-1}\int_0^{\infty} e^{-\lambda^{\frac{\mathfrak{B}}{2}}\zeta} c(\zeta)u_1 d\zeta \\ &\quad + \int_0^{\infty} e^{-\lambda^{\frac{\mathfrak{B}}{2}}\zeta}\varpi(\zeta)\mu(\lambda)d\zeta \end{aligned}$$

. Let $\phi(\theta) = \frac{a}{\theta^a + 1}M_a(\theta^{-a})$, $\theta \in (0,\infty)$. Taking its Laplace

$$\int_0^{\infty} e^{-\lambda^{\theta}}\phi_a(\theta)d\theta = e^{-\lambda^a}. \tag{6}$$

For $a \in (1/2,1)$ using equation (5) we have

$$\begin{aligned}
& \lambda^{a-1}\int_0^\infty c(\zeta)A^{-1}(0)g(u)d\zeta \\
= & \int_0^\infty a(\lambda t)^{a-1}e^{-(\lambda\zeta)}c(\zeta^a)A^{-1}(0)g(u)d\zeta \\
= & \int_0^\infty (-\frac{1}{\lambda})\frac{d}{d\zeta}\Big(\int_0^\infty e^{-\lambda\zeta}\phi(\theta)d\theta\Big)c(\zeta^a)A^{-1}(0)g(u)d\zeta \\
= & \int_0^\infty\int_0^\infty \theta\phi_a(\theta)e^{-\lambda t\theta}c(t^a)A^{-1}(0)g(u)d\theta dt \\
= & \int_0^\infty e^{-\lambda\zeta}\big[\int_0^\infty \phi_a(\theta)c(\frac{\zeta^a}{\theta^a})A^{-1}(0)g(u)d\theta\big]d\zeta \\
= & L\big[\int_0^\infty M_a(\theta)c(\zeta^a\theta)A^{-1}(0)g(u)d\theta\big]d\zeta \\
= & L\big[C_a(t)A^{-1}(0)g(u)\big](\lambda). \qquad (7)
\end{aligned}$$

since $L[H_1(t)](\lambda) = \lambda^{-1}$.
By using Laplace Convolution Theorem

$$\begin{aligned}
\lambda^{-1}\lambda^{a-1}C(\zeta)u_1 d\zeta &= L\big[H_1(\zeta)\big](\lambda).L\big[C_a(\zeta)u_1\big](\lambda) \\
&= L\big[(H_1 * C_a)(\zeta)u_1\big](\lambda). \qquad (8)
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_0^\infty e^{-\lambda a\zeta}S(\zeta)\mu(\lambda)d\zeta \\
= & \int_0^\infty a\zeta^{a-1}e^{(-\lambda a\zeta)}S(\zeta^a)\mu(\lambda)d\zeta \\
= & \int_0^\infty\int_0^\infty a\zeta^{a-\zeta}\phi_a(\theta)e^{-\lambda a\zeta}S(\zeta^a)\mu(\lambda)d\theta d\zeta \\
= & L\Big[\int_0^\infty a\zeta^{a-1}M_a(\theta)S(t^a\theta)d\theta\Big](\lambda).L\Big[f(\zeta,u(\zeta),Gu(\zeta))+Bv(\zeta)\Big](\lambda) \\
= & L\Big[\int_0^\zeta (\zeta-\varpi)^{a-1}K_a(\zeta-\varpi)\big(f(\varpi,u(\varpi),Gu(\varpi))+Bv(\varpi)\big)\Big]. \qquad (9)
\end{aligned}$$

By using equation (6),(7),(8), in (2) we get

$u(\zeta) = C_a(\zeta)A^{-1}(0)g(u) + J_a(\zeta)u_1 + \int_0^{\zeta}(\zeta-\varpi)^{a-1}K_a(\zeta-\varpi)\Big[f\big(\varpi,u(\varpi),Gu(\varpi)\big) + Bv(\varpi)\Big]d\varpi.$

This complete the proof. ■

Lemma 2. (25; 26; 27) Let $B = \{v_n\} \subset C(I,U)$ be countable. If there exist $\varphi \in L^1(I)$ in such a way that $\|v_n(\zeta)\| \leqslant \varphi(\zeta)$ *a.e.* $\zeta \in I$, $n = 1,2,3,...,$ then $\mathfrak{B}\big(\{\int_I v_n(\zeta)dt : n \in \mathbb{N}\}\big) \leqslant 2\int_I \mathfrak{B}(B(\zeta))d\zeta$.

To corroborate our conclusion, for every $h \in C(I,U)$ firstly we take the linear evolution equation non-local problem (LNP).

$$\begin{cases} \mathscr{D}^a u(\zeta) + Au(\zeta) = h(\zeta) \;\; \zeta \in I \\ u(0) = A^{-1}(0)g(u) \\ u'(0) = u_1. \end{cases} \tag{10}$$

For the LNP (10), we get the following outcomes.

Lemma 3. Assume that the conditions

(H0) $|A^{-1}(0)| < \dfrac{1}{N}$ holds. Then LNP (10) has a unique mild solution $x \in C(I,U)$ specified by

$$u(\zeta) = A^{-1}(0)g(u)C_a(\zeta)S + J_a(\zeta)u_1 + \int_0^{\zeta}(\zeta-\varpi)K_a(\zeta-\varpi)h(\varpi)d\varpi \;\; \zeta \in I. \tag{11}$$

where $S = (1 - A^{-1}(0)g(u))^{-1}$.

Proof. From the condition (H0), we get

$\|A^{-1}(0)C_a(\zeta_k)\| \leqslant |A^{-1}(0)|.\|C_a(\zeta_k)\| < 1$

According to the operator spectrum theorem, $S := (1 - A^{-1}(0))^{-1}$ exists as operator and is bounded.

Moreover, by Neumann expression, we acquired

$$\|S\| \leqslant \sum_{m=0}^{\infty} \|A^{-1}(0)C_a(\zeta_k)\|^n = \frac{1}{1-}\|A^{-1}(0)C_a(\zeta_k)\| \leqslant \frac{1}{1-N|A^{-1}(0)|}.$$

We can easily see that LNP (see (9)), (10) has a exclusive mild solution $x \in C(I,U)$ expressed by

$$u(\zeta) = C_a(\zeta)u(0) + J_a(\zeta)u_1 + \int_0^{\zeta} (\zeta - \varpi)^{a-1} K_a(\zeta - \varpi)h(\varpi)d\varpi. \tag{12}$$

From equation (11)

$$\begin{aligned} u(0) &= \frac{1}{(1 - A^{-1}(0)C_a(\zeta_a))}\Big[A^{-1}(0)J_a(\zeta_a)u_1 \\ &\quad + A^{-1}(0)\int_0^{\zeta_a} (\zeta_a - \varpi)^{a-1} K_a(\zeta_a - \varpi)h(\varpi)d\varpi\Big]. \end{aligned}$$

Since $1 - A^{-1}(0)C_a(\zeta_a)$ has a bounded inverse operator S, we acquired

$$u(0) = A^{-1}(0)S\int_0^{\zeta} (\zeta - 1)^{a-1} K_a(\zeta - \varpi)h(\varpi)d\varpi$$

Putting the values we get

$$u(0) = A^{-1}(0)S\int_0^{\zeta} (\zeta - 1)^{a-1} K_a(\zeta - \varpi)h(\varpi)d\varpi. \tag{13}$$

where $S = (1 - A^{-1}(0)g(u))^{-1}$. From equations (12) and (13), we know that function $x \in C(I,U)$ satisfies (11). Conversely, can prove thewe function $x \in C(I,U)$ specified in (11) is a mild solution of the linear evolution equation nonlocal problem (10).
This complete the proof. ■

Controlability results

(H1) An equicontinuous semigroup $\triangle(\zeta)(\zeta \geqslant 0)$ of uniformly bounded linear operator generated by $-A$ in U.

(H2) (i) Define the Linear operator $W : L^2(I,U) \to U$ by

$$Wu = \int_o^{\sigma} (\sigma - s)K_a(\sigma - s)Bv(s)ds.$$

has W^{-1} an inverse operator where the value is taken $L^2(I,U)\ker W$ and $\exists\, N_1 > 0, N_2 > 0$ as two constants in such a way that $\|B\| \leqslant N_1, \|W^{-1}\| \leqslant N_2$,
(ii)$\exists$ a constant $a_1 \in (0,a)$ and a function $Z_w \in N^{\frac{1}{a_1}}(I,\mathbb{R}^+)$ such that
$Z_w(\zeta)\beta(\mathscr{D}) \geqslant \beta(W^{-1}(\mathscr{D})(\zeta)),\ \zeta \in I$ for any countable subset $\mathscr{D} \subset U$.

(H3) The function $f : I \times U \times U \to U$ satisfies,
(i) for a.e $t \in I$ and function $f(\zeta,.,.) : U \times U \to U$ is continuous and the function $f(.,x,y) : I \to U$ strongly measurable for each $(x,y) \in U \times U$,
(ii) for any $\tilde{r} > 0$, $\exists$ a constant $a_2 \in (0,a)$ *and function* $K_{\tilde{r}} \in L^{\frac{1}{a_2}}(I,\mathbb{R}^+)$ such that
$\sup\{\|f(\zeta,x,y)\| : \|x\| \leqslant \tilde{r}, \|y\| \leqslant K^* r'\} \leqslant K_{\tilde{r}}(\zeta),\ \zeta \in I,$
where $K_{\tilde{r}}$ satisfies $\liminf_{\tilde{r}\to\infty} \frac{1}{\tilde{r}} \|K_{\tilde{r}}\|_{L^{\frac{I}{a_2}}} \triangle \tilde{r} < \infty$,
(iii) $\exists\, a_3 \in (0,a)$ and a function $\xi \in L^{\frac{1}{a_3}}(I,\mathbb{R}^+)$ in such a way that
$\beta(f(\zeta,\mathscr{D}_1,\mathscr{D}_2)) \leqslant \xi(\zeta)(\beta(\mathscr{D}_1)+\beta(\mathscr{D}_2))\ \ \zeta \in I$ for any countable subset $\mathscr{D}_1,\mathscr{D}_2 \subset U$.

Let

$$\phi(\zeta) = C_a(\zeta)A^{-1}(0)g(u) + K_a(\zeta)u_1$$

.

Applying the hypothesis $H_1(i)$ for every $x_1 \in U$. We define a control function $v(\zeta) = v(\zeta;x)$ by
$v(\zeta;x) = W^{-1}\left[x_1 - \phi(\sigma) - \int_0^{\sigma} (\sigma - \varpi)^{a-1} K_{\mathfrak{a}}(\sigma - \varpi) f(\varpi,u(\varpi),Gu(\varpi))d\varpi\right](\zeta),\ \zeta \in$

I. For any sake of brevity, we write

$$Q(\zeta;x) = Bv(\zeta;x) + f(\varpi,u(\varpi),Gu(\varpi)).$$

$$\tilde{Q}(x) = \phi(\zeta) + \int_0^{\sigma} (\sigma-\varpi)^{\mathfrak{a}-1} K_{\mathfrak{a}} Q(\varpi,x)\, d\varpi.$$

Now introduce the notations

$E_i = \dfrac{\sigma^{\mathfrak{a}-\mathfrak{a}_i}}{(\mathfrak{a}_i+1)}$, $\mathfrak{a}_i = \dfrac{\mathfrak{a}-1}{1-\mathfrak{a}_i}$ $\quad i = 1,2,3;$

$N_3 = E_1\|I_w\|_{L^{\frac{1}{\mathfrak{a}_1}}}$: $N_4 = E_3\|\xi\|_{L^{\frac{1}{\mathfrak{a}_3}}}$

For every $\tilde{r} > 0$, Let $B_{\tilde{r}} := \{x \in C(I,U) : \|x\|_c \leqslant \tilde{r}\}$. From Lemma (1) and (2),it proceed the following results.

Lemma 4. Suppose that (H2)(i) and (H3)(ii) hold. Then we get

$\|Q(\zeta;x)\| \leqslant N_1N_2\|x_1\| + N_1N_2\|x_2\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{I}{\mathfrak{a}_2}}} + K_{\tilde{r}}(\zeta)$

$\|\widetilde{Q}(x)\| \leqslant \dfrac{NN_1N_2\sigma^{\mathfrak{a}}|A^{-1}(0)|}{\Gamma(\mathfrak{a}+1)(1-N|A^{-1}(0)|)}\bigg(\|x_1\| + \|x_2\|\bigg) + \dfrac{NE_2|A^{-1}(0)|}{\Gamma(\mathfrak{a})(1-N|A^{-1}(0)|)}\bigg(\dfrac{Lub^{a}}{a} + 1\bigg)\|K_{\tilde{r}}\|_{L^{\frac{I}{a_2}}}$

for any $x \in B_{r'}$,where $Lu = \dfrac{NN_1N_2}{\Gamma(\mathfrak{a})(1-N|A^{-1}(0)|)}$.

Proof. For any $\zeta \in I$ and $x \in B_{\tilde{r}}$, from Lemma(1) and Lemma(4), we have

$$\begin{aligned}
\|Bv(\zeta;x)\| &\leqslant N_1N_2\|x_1\| + N_1N_2\|\phi(\zeta)\| + N_1N_2 \\
&\quad \|\int_0^{\sigma}(\sigma-\varpi)^{a-1}K_{\mathfrak{a}}(\sigma\varpi)f(\varpi,u(\varpi),Gu(\varpi))d\varpi\| \\
&\leqslant N_1N_2\|x_1\| + \|x_2\| + \frac{N^2N_1N_2A^{-1}(0)}{\Gamma(a)(1-N|A^{-1}(0)|)} \\
&\quad + \frac{NN_1N_2}{\Gamma(\mathfrak{a})}\int_0^{\sigma}(\sigma-\varpi)^{\mathfrak{a}-1}K_{\mathfrak{a}}(\sigma-\varpi)F_{r'}(\varpi)d\varpi \\
&\leqslant N_1N_2\|x_1\| + N_1N_2\|x_2\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{1}{\mathfrak{a}_2}}}.
\end{aligned}$$

Hence we can observe that
$\|Q(\zeta;x)\| \leqslant N_1N_2\|x_1\|+N_1N_2\|x_2\|+LuE_2\|K_{\tilde{r}}\|_{L^{\frac{1}{\mathfrak{a}_2}}}+K_{\tilde{r}}(\zeta).$
for all $\zeta \in I$ and $x \in B_{\tilde{r}}$. Futhermore, we obtain

$$\begin{aligned}
\|\widetilde{Q}(x)\| &\leqslant \|\phi(\zeta)\|+\frac{|A^{-1}(0)|}{1-N|A^{-1}(0)|}\cdot\frac{N}{\Gamma(\mathfrak{a})}\int_0^{\zeta}(\zeta-\varpi)^{\mathfrak{a}-1}\|Q(\varpi;x)\|d\varpi \\
&\leqslant \|\phi(\zeta)\|+\frac{|A^{-1}(0)|}{1-N|A^{-1}(0)|}\cdot\frac{N}{\Gamma(a)}\int_0^{\zeta}(\zeta-\varpi)^{\mathfrak{a}-1}\big[N_1N_2\|x_1\|+N_1N_2\|x_2\| \\
&+ LuE_2\|K_{\tilde{r}}\|_{L^{\frac{I}{\mathfrak{a}_2}}}+K_{\tilde{r}}(t)\big]d\varpi \\
&\leqslant \frac{NN_1N_2\sigma^{\mathfrak{a}}|A^{-1}(0)|}{\Gamma(\mathfrak{a}+1)(1-N|A^{-1}(0)|)}\Big(\|x_1\|+\|x_2\|\Big) \\
&+ \frac{NE_2|A^{-1}(0)|}{\Gamma(\mathfrak{a})(1-N|A^{-1}(0)|)}\left(\frac{Lu\sigma^{\mathfrak{a}}}{\mathfrak{a}}+1\right)\|K_{\tilde{r}}\|_{L^{\frac{I}{\mathfrak{a}_2}}}.
\end{aligned}$$

This complete the proof. ∎

Define an operator $Y : C(Y;U) \to C(I;U)$ by

$$(Yu)(\zeta)=\widetilde{Q}(x)+\int_0^{\zeta}(\zeta-\varpi)^{\mathfrak{a}-1}K_{\mathfrak{a}}(\zeta-\varpi)Q(\varpi;x)d\varpi,\quad \zeta\in I. \tag{14}$$

Lemma 5. Let (H2)(i) and (H3)(i,ii) hold. Then the operator $Y : B_{\tilde{r}} \to B_{\tilde{r}}$ is continuous proceeded that

$$\frac{N\eta E_2}{\Gamma(a)(1-N|A^{-1}(0)|)}\left[\frac{Lu\sigma^{a}}{a}+1\right]<1. \tag{15}$$

Proof. Firstly, we verify that $Y(B_{\tilde{r}}) \subset B_{\tilde{r}}$ for $\tilde{r}>0$. If this was not the case, $\exists\, x \in B_{\tilde{r}}$

and $\zeta_{\tilde{r}} \in I$ in such a way that $\|(Yu)(t)\| \geqslant \tilde{r}$. By Lemma (1) and (4), we have

$$\begin{aligned}
\tilde{r} &\leqslant \|\widetilde{Q}(x)\| + \|\int_0^{\zeta} (\zeta-\varpi)^{a-1} K_a(\zeta-\varpi) Q(\varpi;x) d\varpi\| \\
&\leqslant \|\widetilde{Q}(x)\| + \frac{N}{\Gamma(a)} \int_0^{t_{\tilde{r}}} (t_{\tilde{r}}-\varpi)^{a-1} \big[N_1N_2\|x_1\| + N_1N_2\|x_2\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{I}{a_2}}} \\
&\quad + K_{\tilde{r}}(\zeta)\|d\varpi\big] \\
&\leqslant \frac{NN_1N_2\sigma^q}{\Gamma(q+1)(1-N|A^{-1}(0)|)} \Big(\|x_1\| + \|x_2\|\Big) \\
&\quad + \frac{NE_2}{\Gamma(a)(1-N|A^{-1}(0)|)} \left(\frac{Lu\sigma^a}{a} + 1\right) \|K_{\tilde{r}}\|_{L^{\frac{I}{a_2}}}.
\end{aligned}$$

Dividing both sides by $\tilde{r}$ and taking lower limit as $\tilde{r} \to +\infty$, we get
$\frac{N\eta E_2}{\Gamma(a)(1-N|A^{-1}(0)|)} \left(\frac{Lu\sigma^a}{a} + 1\right) \geqslant 1.$
Which is a contraction. Hence $Y(B_{\tilde{r}}) \subset B_{\tilde{r}}$ for some $\tilde{r} > 0$.
Now, we will that $Y : B_{\tilde{r}} \to B_{\tilde{r}}$ is continuous. For this target, we suppose that $y_m \to y_0$ in $B_{\tilde{r}}$. We describe $F_m(\varpi) = f(x, y_m(\varpi), Gy_m(\varpi))$ and $F_0(\varpi) = f(x, y_0(\varpi), Gy_0(\varpi))$.By (H3)(i,ii) and Lebesgue dominated convergence theorem, we get
$\int_0^{t_n} (\zeta-\varpi)^{a-1} \|F_m(\varpi) - F_0(\varpi)\| d\varpi \in 0, \quad \zeta \in I(m \to +\infty)$
By the definition of $v(\zeta;x)$, we get
$\|v(\zeta;y_m) - v(\zeta;y_0)\| \leqslant \frac{N^2N_2\|A^{-1}(0)\|}{\Gamma(a)(1-N|A^{-1}(0)|)} \|\phi(\varpi)\| + \frac{NN_2}{\Gamma(a)} \int_o^{\sigma} (\sigma-\varpi)^{a-1} \|F_m(s) - F_0(\varpi)\| d\varpi \to 0 (m \to +\infty).$
Consequently,
$\|Q(\zeta;y_n) - Q(\zeta;y_0)\| \leqslant N_1 \|v(\zeta;y_m) - v(\zeta;y_0)\| + \|F_m(\zeta) - F_0(\zeta)\| \to 0 (n \to +\infty)$
$\|\widetilde{Q}(y_m) - \widetilde{Q}(y_0)\| \leqslant \|\phi(\zeta, y_m) - \phi(\zeta, y_0)\| + \int_0^{\zeta} (\zeta-\varpi)^{a-1} \|Q(\varpi;y_m) - Q(\varpi;y_0)\| d\varpi \to 0 (m \to +\infty).$
Then we get,

$$\|(Yy_m)(\zeta) - (Yy_0)(\zeta)\| \leqslant N\|\widetilde{Q}(y_m) - \widetilde{Q}(y_0)\| + \frac{N}{\Gamma(a)} \int_o^{\zeta} (\zeta-\varpi)^{a-1}$$

$\|Q(\varpi;y_m) - Q(\varpi;y_0)\| d\varpi \to 0 (m \to +\infty)$

Which gives that $Y : B_{\tilde{r}} \to B_{\tilde{r}}$ is continuous. This complete the proof. Now, we can describe the main results of this paper. ■

Theorem 2. Suppose that the hypothesis $(H2)-(H3)$ be satisfied. Then the fractional non-local system (1) is controllable on I provided that (15)

$$\lambda \triangle \frac{2NN_4(1+2K*)}{\Gamma(a)(1-N|A^{-1}(0)|)}[N_3N_5+1] < 1. \tag{16}$$

where $N_5 = \dfrac{2NN_1}{\Gamma(a)(1-N|A^{-1}(0)|)}$.

Proof. We can state an operator $Y : C(I,U) \to C(I,U)$ as (3.1). By taking (5) we can observe that $Y : B_{\tilde{r}} \to B_{\tilde{r}}$ is continuous. We have to justified that Y satisfies Monchs condition. For a particular aims, Let $Y \subset B_{\tilde{r}}$ be countable and $Y \subset \overline{co}(\{0\} \cup Y(\mathscr{D}))$. We will provide that Y is relatively compact. By using the characteritics of measure of noncompactness $\mathfrak{B}$, it is sufficient to demonstrate $\beta(\mathscr{D}) = 0$.
Mainly, we show that $Y(\mathscr{D})$ is continuous on I. For $0 \leqslant \_1 < \delta_2 \leqslant b$, which is denoted by
$S_1 = \|C_a(\delta_2)\widetilde{Q}(\mathscr{D}) - C_a(\delta_1)\widetilde{Q}(\mathscr{D})\|$
$S_2 = \|J_a(\delta_2)\widetilde{Q}(\mathscr{D}) - J_a(\delta_1)\widetilde{Q}(\mathscr{D})\|$
$S_3 = \| \int_0^{\delta_1} [(\delta_1-\varpi)^{a-1} - (\delta_2-\varpi)^{a-1}] K_a(\delta_2-\varpi) Q(\varpi;\mathscr{D}) d\varpi \|$
$S_4 = \| \int_0^{\delta_1} (\delta_1-\varpi)^{a-1} [K_a(\delta_2-\varpi) - K_a(\delta_1-\varpi)] Q(\varpi;\mathscr{D}) d\varpi \|$
$S_5 = \| \int_{\delta_1}^{\delta_2} (\delta_2-\varpi)^{a-1} K_a(\delta_2-\varpi) Q(\varpi;\mathscr{D}) d\varpi \|$.
Then we obtain
$\|(Y(\mathscr{D})(\delta_2)) - (Y(\mathscr{D})(\delta_1))\| \leqslant S_1 + S_2 + S_3 + S_4 + S_5$
From (H1), we can easily see that $S_1, S_2 \to 0$ as $\delta_2 - \delta_1 \to 0$, from Lemma (1) and (5), we have

$$S_3 \leqslant \frac{N\left(N_1N_2\|x_1\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}}\right)}{\Gamma(a)} \int_0^{\delta_1} |(\delta_2-\varpi)^{a-1} - (\delta_1-\varpi)^{a-1}| d\varpi$$
$$+ \frac{N\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}})}{\Gamma(a)} \left(\int_0^{\delta_1} |(\delta_2-\varpi)^{a-1} - (\delta_1-\varpi)^{a-1}|^{\frac{1}{1-a_2}} d\varpi \right)^{1-a_2}$$

$$S_5 \leqslant \frac{N(N_1N_2\|x_1\| + LuE_2\|K_r\|_{L^{\frac{1}{a_2}}})}{\Gamma(a+1)}(\delta_2-\delta_1)^a + \frac{N\|K_r\|}{\Gamma(a)(a_2+1)^{a-a_2}}(\delta_2-\delta_1)^{a-a_2}$$

Which implies that $S_3 \to 0$ *as* $\delta_2 - \delta_1 \to 0$. If $\delta_1 = 0, 0 < \delta_2 \leqslant \sigma$, It is clear that $S_4 = 0$ *for* $\delta_1 > 0$ *and* $\sigma \in (0, \delta_1)$ small enough, We have

$$\begin{aligned} S_4 &\leqslant \| \int_0^{\delta_1-\sigma} (\delta_1-\varpi)^{a-1}[K_a(\delta_2-\varpi) - K_a(\delta_1-\varpi)]Q(\varpi;\mathscr{D})d\varpi\| \\ &+ \int_{\delta_1-\sigma}^{\delta_1} (\delta_1-\varpi)^{a-1}[K_a(\delta_2-\varpi) - K_a(\delta_1-\varpi)]Q(\varpi;\mathscr{D})d\varpi\| \\ &\leqslant \Big[\frac{N\Big(N_1N_2\|x_1\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}}\Big)}{a} + \frac{\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}}(\delta_1^a - \sigma^a)}{(a_2+1)^{1-a_2}}\Big]. \\ &\quad \sup_{\varpi\in[0,\delta_1-\sigma]} \|K_a(\delta_2-\varpi) - K_a(\delta_1-\varpi)\| \\ &+ \frac{2N(N_1N_2\|x_1\| + LuE_2\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}})\sigma^a}{\Gamma(a+1)} + \frac{2N\|K_{\tilde{r}}\|_{L^{\frac{1}{a_2}}}\sigma^{a-a_2}}{\Gamma(a)(a_2+1)^{1-a_2}}. \end{aligned}$$

The supposition (H1) guarantees that $S_4 \to 0$ *as* $\delta_2 - \delta_1 \to 0$ *and* $\sigma \to 0$. Therefore $Y(D)$ is equicontinuous on I.

Now we have to check $\mathfrak{B}(Y(\mathscr{D}))$. From (H2)(ii) and (H3)(iii), we get

$\mathfrak{B}(Bv(\varpi;\mathscr{D})) \leqslant N_4N_5(1+2K^*)K_w(\varpi)\mathfrak{B}(\mathscr{D})$,

$\mathfrak{B}(Q(\varpi;\mathscr{D})) \leqslant N_4N_5(1+2K^*)K_w(\varpi)\beta(D) + (1+2K^*)\xi(\varpi)\mathfrak{B}(\mathscr{D})$

and

$\mathfrak{B}(\widetilde{Q}(\mathscr{D})) \leqslant \frac{2NN_4(1+2K^*)|A^{-1}(0)|}{\Gamma(a)(1-N|A^{-1}(0)|)}[N_3N_5+1]$

for $\varpi \in [0,\zeta], \zeta \in I$. Moreover, we have

$\mathfrak{B}(Y(\mathscr{D})(\zeta)) \leqslant N\mathfrak{B}(\widetilde{Q}(D)) + \frac{2N}{\Gamma(a)} \int_0^{\zeta} (\zeta-\varpi)^{a-1}\mathfrak{B}(Q(\varpi;\mathscr{D}))d\varpi$

$\leqslant \frac{2NN_4(1+2K^*)}{\Gamma(a)(1-N|A^{-1}(0)|)}[N_3N_5+1].\mathfrak{B}(\mathscr{D}) = \lambda\beta(\mathscr{D})$.

It proceeds from $Y(D)$ which continuous and boundedness

$\beta(Y(\mathscr{D})) = \max_{\zeta\in I}\mathfrak{B}(Y(\mathscr{D})(\zeta)) \leqslant \lambda\mathfrak{B}(\mathscr{D})$

Therefore,
$\beta(Y(\mathscr{D})) \leqslant \mathfrak{B}(\overline{co}(\{0\} \cup Y(\mathscr{D}))) \leqslant \mathfrak{B}(Y(\mathscr{D})) \leqslant \lambda \mathfrak{B}(\mathscr{D})$
Since $\lambda < 1$, we get $\mathfrak{B}(\mathscr{D}) = 0$. Then D is relatively compact.
So by 4 , one fix point $x \in B_{\tilde{r}}$ which Y has, it gives a mild solution of the fractional non-local system (1) is s controllable on I and satisfied $x(b) = x_1$.
This summarizes the proof. ■

Remark 1. 1 , relating to a particular nonlocal function, we present another definition of the mild solutions of system (1). According to following newly definition, we state a control function and verify the system (1) of controllability including NCSG. Therefore, in this way we get relevant results exist in (28). Let the supposition (H3) be converted into the form which listed below $(H3)'$. The function $f : I \times U \times U \to U$ satisfied the listed conditions.

(i) for a.e $\zeta \in I$ and function $f(\zeta,.,.) : U \times U \to U$ is equi-continuous and for some $(x,y) \in U \times U$, have a strongly measurable $f(.,x,y) : I \to X$.

(ii) for some $\tilde{r} > 0$, there exist a constant $a_2 \in (0,a)$ *and function* $\tilde{K} \in L^{\frac{1}{a_2}}(I,\mathbb{R}^+)$ such that $\sup\{\|f(\zeta,x,y)\| : \|x\| \leqslant \tilde{r}, \|y\| \leqslant K^*\tilde{r}\} \leqslant \tilde{K}(\zeta), \ \zeta \in I$

(iii) There exists a constant $a_3 \in (0,a)$ and a function $\xi \in L^{\frac{1}{a_3}}(I,\mathbb{R}^+)$ in such a way that
$\beta(f(\zeta,\mathscr{D}_1,\mathscr{D}_2)) \leqslant \xi(\zeta)(\beta(\mathscr{D}_1)+\beta(\mathscr{D}_2)) \ \ \zeta \in I$ for any countable subset $\mathscr{D}_1,\mathscr{D}_2 \subset U$.

Corollary 1. Let the supposition (H0)-(H2) and $(H3)'$ be satisfied. Then given that (15) the fractional non-local system (1) is controllable on I

Remark 2. In supposition $(H3)'$(ii), $\tilde{K}$ of $0 < \tilde{r}$ is independent. In particularly, bounded function is $f(\zeta,x,y)$. Then the inequality (15) automatically takes place due to $\rho = 0$. So, Corollary 1 is most preferable to utilize in application.

4. Application

To emphasize the main result, we suppose the fractional dynamical system of the form

$$\begin{cases} \dfrac{\partial^{\frac{5}{3}}}{\partial t^{\frac{5}{3}}} u(t,y) = \dfrac{\partial u(\zeta,y)}{\partial y} + \dfrac{e^{-2}}{1+e^{\zeta}}\left[u(\zeta,y) + \int_0^{\zeta}(\zeta-s)^2 u(s,y)ds\right] + \omega\kappa(\zeta,y) \quad \zeta \in I \\ u(\zeta,1) = u(\zeta,2) = 0 \ \ ,u(1,y) = \arctan\dfrac{1}{2k^2}g(u) \\ u'(0,y) = u_1(y). \end{cases} \tag{17}$$

Proof. where $0 < \omega$ and $b > m > 0$ are constant, $\kappa : I \times (1,2) \to (1,2)$ is continuous on $I = [0,b]$.
Let $U = X = C([1,2])$ and if $A : \mathscr{D}(A) \subset U \to U$ be defined by
$A\varsigma = -\varsigma' : \varsigma \in \mathscr{D}(A)$
$\mathscr{D}(A) = \{\varsigma \in U, \varsigma(1) = \varsigma(2) = 0\}$.
As we know very clearily that in U an equicontinuous semigroup $\triangle(0 \leqslant \zeta)$ genersted by $-A$ and it is given by
$\triangle(\zeta)\varsigma(s) = \varsigma(\zeta+s)$
for $\varsigma \in U$, $Then\ \triangle(\zeta)(0 \leqslant \zeta)$ is not a compact semigroup in U and $\sup_{\zeta\in I}\|\triangle(\zeta)\| \leqslant 1$.
Define

$$\begin{aligned} u(\zeta)(y) &= u(\zeta,y), \qquad \mathscr{D}^{\frac{5}{3}}u(\zeta)(y) = \frac{\partial^{\frac{5}{3}}}{\partial\zeta^{\frac{5}{3}}}u(\zeta,y) \\ f(\zeta,u(\zeta),Gu(\zeta))(y) &= \frac{\partial u(\zeta,y)}{\partial y} + \frac{e^{-2}}{1+e^{\zeta}}\left[u(\zeta,y) + \int_0^{\zeta}(\zeta-s)^2 u(s,y)ds\right] \\ u(\zeta)(y) &= \kappa(\zeta,y), \qquad A^{-1}(0) = \arctan\frac{1}{2k^2}. \end{aligned}$$

Then for some $x \in B_{\tilde{r}} := \{x \in C(I,U); \|x\|_c \leqslant\},\ \zeta \in I$, we have

$$
\begin{aligned}
\|f(\zeta,u(\zeta),Gu(\zeta))(y)\| &\leqslant \frac{\partial u(\zeta,y)}{\partial y} + \frac{e^{-2}}{1+e^{\zeta}}\left[\|u(\zeta,y)\| + \int_0^{\zeta} \|(\zeta - s)^2 u(s,y)\| ds\right] \\
&\leqslant \frac{(3+b^3)e^{-2\zeta}\tilde{r}}{3(1+e^{\zeta})} \\
&\leqslant \frac{(3+\sigma^3)\tilde{r}}{6}.
\end{aligned}
$$

It is mmostly knowning that the supposition (H_3) takes place for $\gamma = \dfrac{3+\sigma^3}{6}$ $and\ \xi(\zeta) = \frac{1}{2}\ for\ all\ \zeta \in I$. From
$\|A^{-1}(0)\| \leqslant \|\arctan \dfrac{1}{2k^2}\| = \dfrac{\pi}{4} < 1.$
As the supposition (H_0) takes place. For $y \in (1,2)$, the operator W is described by

$$
(W(\zeta))(y) = \int_0^{\sigma} (\sigma - s)^{\frac{-1}{4}} K_{\frac{3}{4}}(\sigma - s) Bv(s) ds.
$$

where
$\{C_{\frac{5}{3}(t)}\}_{0\geqslant\zeta}$, $\{K_{\frac{5}{3}(t)}\}_{0\geqslant\zeta}$ and $\{J_{\frac{5}{3}(\zeta)}\}_{0\geqslant\zeta}$ are defined by

$$
\begin{aligned}
C_{\frac{5}{3}(\zeta)\varsigma(s)} &= \int_0^{\infty} M_{\frac{5}{3}}(\theta)\varsigma(\zeta^{\frac{5}{3}}\theta + s))d\theta \\
K_{\frac{5}{3}}(\zeta)\varsigma(s) &= \frac{5}{3}\int_0^{\infty} \theta M_{\frac{5}{3}}(\theta)\varsigma(\zeta^{\frac{5}{3}}\theta + s))d\theta \\
J_{\frac{5}{3}}(\zeta)\varsigma(s) &= \int_0^{\zeta} C_{\frac{5}{3}}(s)\varsigma(\zeta^{\frac{5}{3}}\theta + s))ds.
\end{aligned}
$$

where $M_{\frac{5}{3}}(\theta) = \dfrac{5}{3}\theta^{\frac{-8}{5}}\rho_{\frac{5}{3}}(\theta^{\frac{-3}{5}})\ \ for\ \ 0 < \theta < \infty$ and $\rho_{\frac{5}{3}}(\theta)$ is given by

$$
\rho_{\frac{5}{3}} = \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\theta^{\frac{-5m}{3}-1}\frac{\Gamma(m\frac{5}{3}+1)}{m!} sin(\frac{5m\pi}{3}),\ \ \theta \in (0,\infty)
$$

. ■

Conclusion

In summary, our study focused on examining the precise controllability of non-local Cauchy problems related to fractional integro-differential evolution equations in Banach spaces with non-compact semigroups and non-local functions. To achieve this, we established a relevant definition for mild solutions, utilized a specific type of non-local function, and employed the Mönch fixed point theorem to demonstrate exact controllability in the case of non-compact semigroups. As a result, our findings demonstrate the effectiveness of these theoretical outcomes.

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