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Optimal Control of Fractional Integro-differential Evolution Equation with Infinite Delay of order (1; 2)

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Abstract In this paper, we will examine the facts in the image of infinite time delay on the mild solution's solvability and fractional integro-differential Evolution Equation of Banach space's optimal control choosing order $\delta \in (1,2)$. For investigation of mild solution of the system, continuation, uniqueness and the existence of the solution with the help of grown-wall inequality will be discussed shortly. After constructing the lagrange problem, the demonstration of actuality of optimal control of fractional integro delay system is being proceed. And at the end we will discuss the regarding result by an adequate example.

Keywords: Fractional integro-differential evolution equations; Mild solution; Existence; Continuous dependence; Optimal controls
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1. Introduction

The most budding branch of applied mathematics is fractional calculus. A lot of developments and upbeat work has been done in this category. No one can deny the impulse of fractional calculus and differential equation but it leads to become more

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consequential day by day. This work became up rear after the liaison of L.Hospital and Leibnitz. In the historical record the systematic investigation on optimal control motors has been done in 1960. Podlubny,s monographs (1), Kilbas et al. (2), Zhou (3; 4) and papers (5; 6; 7; 8; 9) and the references therein show that ordinary and differential equations containing fractional derivatives have significantly advanced in the modern era The attracted work has been done on the existence of being mild solution of order $\delta \in (1,2)$ for the fractional differential and integro-differential equations in contemporary. The two fractional evolution systems are taken along Riemann-liouville derivative with the help of resolvent family by Li et al. (10). The being and uniqueness of mild solution for nonlocal fractional differential equation is deliberate by(11) by installing resolvent family. Li (12), (14) scrutinize the constancy of mild solution for fractional abstract cauchy problems with the help of analytical solution operators. As well as, Kian and Yamamoto (13) scrutinize the existence and Strichartz evaluates the solutions for semi linear fractional wave equations by the method of eigenvalue elaboration in bounded domain. Many of the prime controllability and fascinating results on the fractional differential systems initiate in (15) with order $\delta \in (0,1)$ and in (16; 17) with order $\delta \in (1,2)$.

At present fractional evolution equation,s optimal control has striking the role models in different aspects of Science, Engineering and Economy. This field makes more advancements exceptionally in control doctrine, biological science, physical science, electronic media, elasticity, electromagnetic, electro dynamic process etc. It is owing to this stipulate of optimal power of fractional evolution equation in allied area of cram, conjecture and its relevance has been improved to a enormous level. Loads of analyzers are geared up to hop away ahead it to carve up their involvement in this fastidious branch. A fabulous work has been done in the category of optimal controls and fractional functional evolution equation by Wang et al. (18) in 2011. He investigate after choosing α-norm for mild solution's existence of semi linear fractional functional evolution equation, optimal control. The examination for infinite dimensional space on fractional finite time delay system of evolution equation and optimal controls has been done by Wang et al. in 2011 (19). His concluding explore leads to the Lagrange problem. Due to his ardent concentration, the same author (20), expand this gate of knowledge to the class of time optimal control of non linear fractional integro-differential controlled system associated with analytic semigroup in Banach space. Moreover the same author also work on fractional integro-differential evolution system,s solvability and optimal controls with infinite delay in 2012(21) choosing order$(0,1)$. In 2017, Jun Du et al. (22) established the controllability for a rare form of nonlocal and arbitrarily delayed fractional neutral integro-differential evolution equations. In the course of time, some researches subsidize tHeir attempts to increase order by $(1,2)$. For instance, many researchers analysed fractional equations of Caputo and Riemann-Liouville's Sobolev typ with orders of $(1,2)$ (23). Consider the system of nonlinear fractional integro-differential evolution system as follows:

$$\begin{cases} {}_0^cD_\xi^\delta u(\xi) = Eu(\xi) + F(\xi,u_\xi, \int_0^\xi f(\xi,y,u_y)dy) + G(\xi)x(\xi),\ \xi \in \tau = [0,T] \\ u(\xi) = \phi(\xi) \in \mathscr{A}, u'(\xi) = \phi'(\xi) \in \mathscr{A}, -\infty \leqslant \xi \leqslant 0. \end{cases} \tag{1}$$

where ${}_0^cD_\xi^\delta$ is Caputo fractional derivative of order $\delta \in (1,2)$, operator$E : D(E) \to \varsigma$ is cosine family $\{\mathscr{C}(\xi)\}_{\xi \geqslant 0}$ on a Banach space's infinitesimal generator ς which is strongly continuous, F and f are ς-value functions specified later, G is a linear operator from S into ς, The chronicles define u_ξ: $]-\infty,0] \to \varsigma$ by $u_\xi = u(\xi + y)$, related to any abstract space $\mathscr{A}$. The structure of this paper is as follows. The notations and helpful ideas for fractional integro-differential and the cosine family are recalled in Section 2. An operator will be introduced in this paper which differs from the preceding one in the families of operators. Before moving on to the next stage of the study, it should be noted that this operator is continuous and linearly bounded. Additionally, the analysis demonstrate problem is mildly solvable, that the mild solution is distinct, facts in Section 4 are continuously dependent. In section 5, we will demonstrate that for Lagrange problem, there are fractional optimal controls.

2. Preliminaries

Consider the two Banach spaces ς and S with the norms $|\cdot|_\varsigma$ and $|\cdot|_S$, commonly. $\pounds(\varsigma,S)$ represent linear operator's space from ς to S set up with norm $\|\cdot\|_{\pounds(\varsigma,S)}$. Specially , when $\varsigma = S$, then $\pounds(\varsigma,S) = \pounds(\varsigma,S) = \pounds(\varsigma)$ and $\|\cdot\|_{\pounds(\varsigma,S)} = \|\cdot\|_{\pounds(\varsigma,\varsigma)} = \|\cdot\|_{\pounds(\varsigma)}$. Let a Banach space $C(]-\infty,a^o],\varsigma), a^0 \geqslant 0$ of continuous functions from $]-\infty,a^0]$ to ς with usual sup-norm. We represent $C(]-\infty,a^0],\varsigma)$ frequently by $C_{-\infty,a^0}$ and its norm by $\|\cdot\|_{-\infty,a^0}$. If $a^0=\tau$ we represent this space by $C_{-\infty,\tau}$ and its norm by $\|\cdot\|_{-\infty,\tau}$. If $a^0=0$, we represent this space by $C_{-\infty,o}$ and its norm by $\|\cdot\|_{-\infty,0}$. Surely, for any $u\in C_{-\infty,\tau}$ and $\xi\in\tau$, define $u_\xi(s)=u(\xi+s)$ for $-\infty\leqslant s\leqslant 0$, then $u_\xi\in C_{-\infty,0}$. We denote G by $R(\mu,G)=(\mu I-G)^{-1}\in\mathfrak{L}(\varsigma)$ as a resolvent set.

Remembering the definitions and characteristics of fractional calculus, we suggest to see (1; 2).

The fractional integral of order $\delta\in\mathbb{R}_+$ with the lower limit zero for a function u is defined as

$$I_{0+}^{\delta}u(\xi)=(g_\delta*u)(\xi)=\frac{1}{\Gamma(\delta)}\int_0^{\xi}(\xi-y)^{\delta-1}u(y)dy,$$

The right-wards defined on $[0,\infty)$, here *Gamma* is Euler gamma function and the sign $*$ signifies convolution.

$$g_\delta(\xi)=\frac{\xi^{\delta-1}}{\Gamma(\delta)},\quad \text{if } \xi>0;\quad g_\delta(\xi)=0,\quad \text{if } \xi\leqslant 0.$$

In case $\delta=0$, we signify $g_0(\xi)=\delta(\xi)$, For a function $u:[0,\infty)\to\mathbb{R}$, the Riemann-Liouville fractional derivative of order $\delta\in\mathbb{R}_+$ is defined as

$${}_0^LD_\xi^\delta u(\xi)=\frac{d^n}{d\xi^n}(g_{n-\delta}*u)(\xi),\quad \xi\geqslant 0, n-1<\delta<n,$$

For function $u:[0,\infty)\to\mathbb{R}$, Caputo derivative of order $\delta\in\mathbb{R}_+$ is defined as

$${}_0^cD_\xi^\delta u(\xi)={}_0^LD_\xi^\delta\Big(u(\xi)-\sum_{k=0}^{n-1}\frac{w^{(k)}(0)}{k!}\xi^k\Big),\quad \xi\geqslant 0, n-1<\delta<n.$$

Lemma 1. (see(29, Lemma1.2)). Suppose that $u\in C_{0,\tau}$ satisfies the following inequality:

$$|u(\xi)|\leqslant a^0+b^0\int_0^{\xi}(\xi-s)^{b-1}\|u(s)\|_{-r,0}ds,\ \xi\in\tau, u(\xi)=\phi(\theta),\ -\infty\leqslant\xi\leqslant 0,$$

constant $a^0, b^0 \geqslant 0$. A positive constant $\mathscr{M}^*$ exists there which is independent of a^0 in such a way $|u(\xi)| \leqslant \mathscr{M}^*(a^0)$, for all $\xi \in \tau$.

Lemma 2. (see(30, Lemma 2.8)). A function $W : \tau \to \varsigma$ is Bochner integrable and measurable, if $\|W\|$ is Lebesque integrable.

Lemma 3. (see(31, Problem 23.9)). For every $\phi \in L^a(\tau,\varsigma)$ with $1 \leqslant a < +\infty$,

$$\lim_{h\to 0}\int_0^\tau |\phi(\xi+h)-\phi(\xi)|^a d\xi = 0$$

here , $\phi(s) = 0$ for $s \notin \tau$.

Definition 1. If $\mathscr{C}(0) = I$, $\mathscr{C}(y+\xi)+\mathscr{C}(y-\xi) = 2\mathscr{C}(y)\mathscr{C}(\xi)$ for all $y,\xi \in \geqslant 0$ and $\mathscr{C}(\xi)x$ is equicontinuous in ξ on $[0,\infty)$ for every constant point $x \in \varsigma$ then bounded linear operators $\{\tau(\xi)\}_{\xi\geqslant 0}$'s family mapping the Banach space ς into itself only .

Cosine family and continuous sine family defined by $\{\mathscr{S}(\xi)\}\xi \geqslant 0$:

$$\mathscr{S}(\xi)x = \int_0^\xi \mathscr{C}(y)xdy, \quad x \in \varsigma,\, \zeta \geqslant 0.$$

Definition 2. The linear operator E, is defined as

$$D(E) = u \in \varsigma; \lim_{\xi\to 0} E_\xi u = \lim_{\xi\to 0}\frac{Y(\xi)u-u}{\xi} exists;$$

consider the infinitesimal generator E of continuous cosine family cosine family $\{\mathscr{T}(\xi)\}_{\xi\geqslant 0}$ in Banach space ς. Let $\mathscr{M} = sup_{\xi\in\tau}\{\|Y(\xi)\|_{(\varsigma)}\}$ be finite number. Suppose $\mathscr{D}(\tau,\varsigma)$, be continuous function's Banach space from τ to ς with usual supreme norm $\|u\|_{\mathscr{D}} = sup_{\xi\in\tau}\{\|u(\xi)\|\|\}$.

Now we use the fundamental definition of the phase space $\mathscr{A}$ introduced by Kato and Hale (35). Suppose $\mathscr{A}$ be function's linear face mapping $]-\infty,0]$ to ς having semi norm $\|\cdot\|_{\mathscr{A}}$ and satisfying the axioms:

(P1) If $u :]-\infty,0] \to \varsigma$, such that $u_0 \in \mathscr{A}$ then for each $\xi \in \tau$, the listed conditions satisfied:

(i) u_ξ is in $\mathscr{A}$,

(ii) $\|u(\xi)\| \leqslant H^*\|u_\xi\|_{\mathscr{A}}$,

(iii) $\|u_\xi\|_{\mathscr{A}} \leqslant K^*(\xi)sup\|u(y)\| : 0 \leqslant y \leqslant \xi\} + P(\xi)\|u_0\|_{\mathscr{A}}$
where $H^* \geqslant 0$ is a constant, $K^* : \tau \to [0,+\infty]$ is continuous,$P : [0,+\infty] \to [0,+\infty]$ is bounded and H^*,K^*,P are independent of u.

(P2) For the function u in $(P1)$,x_ξ is a $\mathscr{A}$-valued function in τ.

(P3) The $\mathscr{A}$ is complete.

Define $\mathscr{A}\mathscr{D} := u :]-\infty,0] \to \varsigma, u|_{]-\infty,0]} \in \mathscr{A}$ and $u|_\tau \in \mathscr{D}(\tau,\varsigma)$ and let $\|\cdot\|_{\mathscr{A}\mathscr{D}}$ be the semi norm in $\mathscr{A}\mathscr{D}$ define by $\|u\|_{\mathscr{A}\mathscr{D}} = \|u_0\|_{\mathscr{A}} + sup_{y\in\tau}\{\|u(t\xi)\|\}$. it is not difficult to understand $\mathscr{A}\mathscr{D},\|\cdot\|_{\mathscr{A}\mathscr{D}}$ is a Banach space. Describe a set $\mathscr{A}\mathscr{D}^0 := s \in \mathscr{A}\mathscr{D} : s_(0) = 0 \in \mathscr{A}$ *and let* $\|\cdot\|_{\mathscr{A}\mathscr{D}^0} := \|s_0\|_{\mathscr{A}} + \sup_{y\in\tau}\{\|s(\xi)\|\}$ and $\mathscr{A}\mathscr{D}^0,\|\cdot\|_{\mathscr{A}\mathscr{D}^0}$ is a Banach space.

3. Existence and Uniqueness

Here, we establish system's existence and uniqueness (1). Let E be infinitesimal generator of a strongly continuous cosine family of $\{C(\xi)\}_{\xi\geqslant 0}$ which is uniformly bounded linear operators, i.e., that $M \geqslant 1$ exists such that $\|C(\xi)\|_{\mathfrak{L}(\varsigma)} \leqslant M$, for $\xi \geqslant 0$. For convenience, let $n = \delta/2$ with $\delta \in (1,2)$. The following assumptions are listed:

(H1) $F : \tau \times \mathscr{A} \times \varsigma \to \varsigma$ satisfies:

(i) For $\xi \in \tau F$ is measurable.

(ii) For arbitrary $\varpi_1, \varpi_2 \in \mathscr{A}$, $\Psi_1, \Psi_2 \in \varsigma$ satisfying $\|\varpi_1\|_{\mathscr{A}}$, $\|\varpi_2\|_{\mathscr{A}}$, $\|\Psi_1\|$, $\|\Psi_2\| \leqslant \rho$, $\exists L_F(\rho) > 0$ in such a way that

$$\|F(\xi,\varpi_1,\Psi-1) - F(\xi,\varpi_2,\Psi_2\| \leqslant L_F(\rho)(\|\varpi_1 - \varpi_2\|_{\mathscr{A}} + \|\Psi_1 - \Psi_2\|)$$

for all $\xi \in \tau$;

(iii) There exists a constant $a_F^0 > 0$ in such a way that $\|F(\xi,\varpi,\Psi)\| \leqslant a_F^0(1 + \|\varpi\|_{\mathscr{A}} + \|\Psi\|)$ for all $\varpi \in \mathscr{A}, \Psi \in \varsigma$ $\xi \in \tau$.

(H2) $f : D := \{(\xi,y) \in \tau \times \tau | 0 \leqslant y \leqslant \xi\} \times \mathscr{A} \to \varsigma$ satisfies:

(i) For $(\xi,y) \in D$, f is continuous

(ii) For any $\varpi_1,\varpi_2 \in \mathscr{A}$ and $(\xi,y) \in D$ satisfying $\|\varpi_1\|_{\mathscr{A}},\|\varpi_2\|_{\mathscr{A}} \leqslant \rho$, there exist a $L_f(\rho) > 0$ such that
$\|f(\xi,y,\varpi_1) - f(\xi,s,\varpi_1)\| \leqslant L_f(\rho)\|\varpi_1 - \varpi_2\|_{\mathscr{A}}$

(iii) $M_f > 0$ exists there in such a way
$\|f(\xi,y,\varpi)\| \leqslant M_f(1+\|\varpi\|)$ for all $\varpi \in \mathscr{A}$

(H3) Suppose S be separable reflexive Banach from which the control x takes the values. Operator $\mathscr{A} \in L_\infty(\tau,L(S,\varsigma))$.

(H4) Multi-valued map $X(\cdot) : \tau \to 2^S\backslash\Omega$ has bounded, closed and convex values, $X(\cdot)$ is graph measurable and $X(\cdot) \subseteq \Phi$, here Φ is bounded set in S.

A measurable set is described by

$$X_{a^0d^0} = \{w(\cdot) : \tau \to S \; strongly \; measurable. \; s(\xi) \in X(\xi)a.e\}$$

. Obviously, $X_{a^0d^0} \neq \Omega$, and $X_{a^0d^0} \subset L^m(\tau,W)(1 < m < +\infty)$ is closed, convex and bounded. Obviously $Gx \in L^m(\tau,\varsigma)$ for all $x \in X_{a^0d^0}$

(see (2)), we can see that equation (1) has the representation given below

$$u(\xi) = u_0 + u_1\xi + \frac{1}{\Gamma(\delta)}\int_0^\xi (\xi - y)^{\delta-1}[Eu(y) + F(y,u_y,\int_0^\xi f(y,s,u_s)ds) + G(y)x(y)]dy \tag{2}$$

$\xi \in \tau$ the right-ward of the equation takes place. Probability density function $\varpi_n(\theta)$ will be used which is defined on $]0,\infty[$ as

$$\begin{aligned} \varpi_n(\theta) &= \frac{1}{nq\theta^{(1+1/n)}}\vartheta_n(\theta^{-1/n}) \geqslant 0, n \in (0,1) \\ \vartheta_n(\theta) &= \frac{1}{\Pi}\sum_{q=1}^{\infty}(-1)^{q-1}(\theta)^{-nq-1}\frac{\Gamma(qn+1)}{q!}sin(q\Pi n). \end{aligned} \tag{3}$$

Lemma 4. If the formula (2) takes place, then for $\xi \in \tau$, $n = \delta/2$

$$u(\xi) = \S_n(\xi)u_0 + \kappa_n(\xi)u_1 + \int_0^\xi (\xi - y)^{n-1} M_n(\xi - y) F(y, u_y, \int_0^\xi f(y,s,u_s)ds)dy$$
$$+ \int_0^\xi (\xi - y)^{n-1} M_n(\xi - y) G(y) x(y) dy, \tag{4}$$

where
$\S_n(\xi) = \int_0^\infty \varpi_n(\theta)\mathscr{C}(\xi^n\theta)d\theta$, $\kappa_n(\xi) = \int_0^\xi \S_n(s)ds$, $M_n(\xi) = n\int_0^\infty \theta\varpi_n(\theta)\S(\xi^n\theta)d\theta$.

Proof. For $\mu > 0$. we apply Laplace transforms on (2).

$$J(\mu) = \frac{1}{\mu}u_0 + \frac{1}{\mu^2}u_1 + \frac{1}{\mu^\delta}EJ(\mu) + \frac{1}{\mu^\delta}w(\mu) + \frac{1}{\mu^\delta}z(\mu)$$

here $J(\mu) = \int_0^\infty e^{-\mu y}u(y)dy$, $w(\mu) = \int_0^\infty e^{-\mu y})F(y, u_y, \int_0^\xi f(y,s,u_s)ds))dy$,
and $z(\mu) = \int_0^\infty e^{-\mu y}G(y)x(y)dy$ this implies

$$(\mu^\delta I - E)J(\mu) \quad = \quad \mu^{\delta-1}u_0 + \mu^{\delta-2}u_1 + w(\mu) + z(\mu)$$

Therefore, by virtue of the link among cosine function and resolvent, i.e., for $Re\mu > 0$,

$$\mu R(\mu^2; E)x = \int_0^\infty e^{-\mu\xi}\mathscr{C}(\xi)xd\xi, \quad R(\mu^2; A)x = \int_0^\infty e^{-\mu\xi}\S(\xi)xd\xi, \quad x \in \varsigma,$$

we first have

$$\begin{aligned} J(\mu) &= \mu^{\delta-1}(\mu^\delta I - E)^{-1}u_0 + \mu^{\delta-2}(\mu^\delta I - E)^{-1}u_1 \\ &\quad + (\mu^\delta I - E)^{-1}w(\mu) + (\mu^\delta I - E)^{-1}z(\mu) \\ &= \mu^{\frac{\delta}{2}-1}\int_0^\infty e^{-\mu^{\frac{\delta}{2}}y}\mathscr{C}(y)u_0dy + \mu^{\frac{\delta}{2}-2}\int_0^\infty e^{-\mu^{\frac{\delta}{2}}y}\mathscr{C}(y)u_1dy \\ &\quad + \int_0^\infty e^{-\mu^{\frac{\delta}{2}}y}\S(y)w(\mu)dy + \int_0^\infty e^{-\mu^{\frac{\delta}{2}}y}\S(y)z(\mu)dy \end{aligned}$$

As $n = \frac{\delta}{2} \in (1/2,1)$, so we take

$$\begin{aligned} y(\mu) &= \mu^{n-1}\int_0^{\infty} e^{-\mu^n y}\mathscr{C}(y)u_0 dy + \mu^{-1}\mu^{n-1}\int_0^{\infty} e^{-\mu^n y}\mathscr{C}(y)u_1 dy \\ &\quad + \int_0^{\infty} e^{-\mu^n y}\S(y)w(\mu)dy + \int_0^{\infty} e^{-\mu^n y}\S(y)z(\mu)dy \end{aligned} \tag{5}$$

Suppose probability density function which is one sided in (3) then its Laplace transform is given by

$$\int_0^{\infty} e^{-\mu\theta}\vartheta_n(\theta)d\theta = e^{-\mu^n}, \quad n \in]0,1[\tag{6}$$

By using (5) and (6).

$$\begin{aligned} \mu^{n-1}\int_0^{\infty} e^{-\mu^n y}\mathscr{C}(y)u_0 dy &= \int_0^{\infty} \mu^{n-1}e^{-(\mu\xi)^n}\mathscr{C}(\xi^n)n\xi^{n-1}u_0 d\xi \\ &= \int_0^{\infty} n(\mu\xi)^{n-1}e^{-(\mu\xi)^n}\mathscr{C}(\xi^n)u_0 d\xi \\ &= \int_0^{\infty} \frac{-1}{\mu}\frac{d}{d\xi}(e^{-(\mu\xi)^n})\mathscr{C}(\xi^n)u_0 d\xi \\ &= \int_0^{\infty}\int_0^{\infty} \frac{-1}{\mu}\frac{d}{d\xi}(e^{-\mu\xi\theta}\vartheta_n(\theta))\mathscr{C}(\xi^n)u_0 d\theta d\xi \\ &= \int_0^{\infty} e^{-\mu\xi}\int_0^{\infty} \vartheta_n(\theta)\mathscr{C}(\frac{\xi n}{\theta^n})u_0 d\theta d\xi \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty} e^{-\mu\xi}\int_0^{\infty} \frac{1}{n\theta^{(1+1/n)}}\vartheta_n(\theta^{-1/n})\mathscr{C}(\xi^n\theta)u_0 d\theta d\xi \\ &= \int_0^{\infty} e^{-\mu\xi}\int_0^{\infty} \varpi_n(\theta)\mathscr{C}(\xi^n\theta)u_0 d\theta d\xi \\ &= \int_0^{\infty} e^{-\mu\xi}[\S_n(\xi)u_0]d\xi \\ &= L^*[\S_n(\xi)u_0](\mu) \end{aligned} \tag{7}$$

Moreover, as $L^*[g_1^*(\xi)](\mu) = \mu^{-1}$, by using theorem of Laplace convolution, we have

$$\mu^{-1}\mu^{n-1}\int_0^\infty e^{-\mu^n y}\mathscr{C}(y)u_1 dy = L^*[g_1^*(\xi)](mu).L^*[\S_n(\xi)u_1](mu) = L^*[(g_1^* * \S_n)(\zeta)u_1](\mu) \tag{8}$$

Similarly

$$\begin{aligned}
&\int_0^\infty e^{-\mu^n y}\S(y)w(\mu dy \\
&= \int_0^\infty e^{-\mu\xi}\left[n\int_0^\xi\int_0^\infty \vartheta_n(\theta)\S(\frac{(\xi-y)^n}{\theta^n})F(y,u_y,\int_0^\xi f(y,s,u_s)ds))dy\frac{(\xi-y)^{n-1}}{\theta^n}d\theta dy\right]d\xi \\
&= L^*\left[n\int_0^\xi(\xi-s)^{n-1}\int_0^\infty \vartheta_n(\theta)\S(\frac{(\xi-y)^n}{\theta^n})F(y,u_y,\int_0^\xi f(y,s,u_s)ds))dy\frac{1}{\theta^n}d\theta dy\right](\mu) \\
&= L^*\left[\int_0^\xi(\xi-y)^{n-1}M_n(\xi-y)F(y,u_y,\int_0^\xi f(y,s,u_s)ds))dy\right](\mu) \qquad (9)
\end{aligned}$$

And also similarly

$$\int_0^\infty e^{-\mu^n y}\S(y)z(\mu)dy \quad = \quad L^*\left[\int_0^\xi(\xi-y)^{n-1}M_n(\xi-y)G(y)x(y)dy\right](\mu) \tag{10}$$

After combining equations (7), (8), (9)and (10) we get our result i.e. (4). and the proof is completed. ∎

Definition 3. For any $x \in \hat{L}^m(\tau,S)$, if there exist $C = C(x) > 0$ and $u \in C(\tau,\varsigma)$ such that

$$\begin{aligned}
u(\xi) &= \S_n(\xi)u_0 + \kappa_n(\xi)u_1 + \int_0^\xi(\xi-y)^{n-1}M_n(\xi-y)F(y,u_y,\int_0^\xi f(y,s,u_s)ds))dy \\
&+ \int_0^\xi(\xi-y)^{n-1}M_n(\xi-y)G(y)x(y)dy \qquad (11)
\end{aligned}$$

The system (1) is mildly solvable with respect to x on $[0,\tau]$.

Lemma 5. The operator $\S_n$, κ_n and M_n have the listed characteristics:

(i) for any fixed $\xi \geqslant 0$, $\S_n(\xi)$,$\kappa_n(\xi)$ and $M_n(\xi)$ are linear and bounded operators, i.e for any $u \in \varsigma$.

$$|\S_n(\xi)u| \leqslant P|u|,\ |\kappa_n(\xi)u| \leqslant P\xi|u|,\ |M_n(\xi)w| \leqslant \frac{P}{\Gamma(2n)}\xi^n|u|.$$

(ii) $\{\S_n(\xi)\}_{\xi\geqslant 0}$,$\{\kappa_n(\xi)\}_{\xi\geqslant 0}$ and $\{M_n(\xi)\}_{\xi\geqslant 0}$ are strongly continuous.

(iii) For every $\xi > 0$, $\S_n(\xi)$, $\kappa_n(\xi)$ and $M_n(\xi)$ are also compact operators if $T(\xi)$ is compact.

Proof. Proof can be seen in (36) and (37). ■

Lemma 6. Consider $\phi(0)$,$\phi'(0) \in \varsigma$, (H1)(iii), (H2)(iii) holds. Also suppose system 1 is mildly solvable with respect to $x \in X_{a^0 d^0}$ on $[-\infty,T]$, then their exist constant $\rho > 0$ such that $\|u(\xi)\| \leqslant \rho$; for all $\xi \in \tau$

Proof. As the system 1 is mildly solvable with respect to $x \in X_{a^0 d^0}$ on $]-\infty,T]$, then by Definition 3.1, we can suppose u is mildly solvable of the system 1 with respect to x on $[-\infty,T]$, then u satisfies 11. Let $u(\xi) = s(\zeta) + \tilde{\phi}(\xi)$ where $\tilde{\phi} :]-\infty,T] \to \varsigma$ be a function given by

$$\tilde{\phi}(\xi) = \begin{cases} \phi(\xi), -\infty < \xi \leqslant 0, \\ \phi'(\xi), -\infty < \xi \leqslant 0, \\ \S_n(\xi)\phi(0), \xi \in \tau. \end{cases} \tag{12}$$

surely, u satisfies 11 iff

$$\begin{cases} s_0 = 0, -\infty < t \leqslant 0, \\ s(\xi) = \S_n(\xi)s_0 + \kappa_n(\xi)s_1 + \int_0^\zeta (\xi - y)^{n-1} M_n(\xi - y) F(y, s_y + \tilde{\phi}_y, \\ \int_0^\xi f(y, J, s_J + \tilde{\phi}_J)dJ))dy + \int_0^\zeta (\xi - y)^{n-1} M_n(\xi - y) G(y)x(y)dy, \xi \in \tau. \end{cases} \tag{13}$$

For $t \in \tau$ gives the result

$$\|s(\xi)\| \leqslant \|\S_n(\xi)s_0\| + \|\kappa_n(\xi)s_1\| + \int_0^{\xi}(\zeta - y)^{n-1}\|M_n(\xi - y)F(y,s_y + \tilde{\phi}_y,$$
$$\int_0^{\xi} f(y,J,s_J + \tilde{\phi}_J)dJ)\|dy + \int_0^{\xi}(\xi - y)^{n-1}||M_n(\xi - y)G(y)x(y)\|dy$$
$$\leqslant \|\S_n(\xi)s_0\| + \|\kappa_n(\xi)s_1\| + \frac{nP}{\Gamma(1+n)}\int_0^{\xi}(\xi - y)^{n-1}\bar{a}_F(1 + \|s_y + \tilde{\phi}_y\|_{\mathscr{A}}$$
$$+ P_f T(1 + \|s_J + \tilde{\phi}_{\xi}\|_{\mathscr{A}}))dy + \frac{nP\|G\|_{\infty}}{\Gamma(1+n)}\int_0^{\zeta}(\xi - y)^{n-1}\|x(y)\|_s dy.$$
$$\leqslant P\|s_0\| + P^*\|s_1\| + \frac{nP}{\Gamma(1+n)}\int_0^{\xi}(\zeta - y)^{n-1}\bar{a}_F(1 + \|s_y + \tilde{\phi}_y\|_{\mathscr{A}} + P_f T$$
$$(1 + \|s_J + \tilde{\phi}_{\xi}\|_{\mathscr{A}}))dy + \frac{nP\|G\|_{\infty}}{\Gamma(1+n)}\int_0^{\xi}(\xi - y)^{n-1}\|x(y)\|_s dy$$
$$\leqslant \bar{a} + \frac{(\bar{a}_F)nP(1 + P_f T)}{\Gamma(1+n)}\int_0^{\xi}(\xi - y)^{n-1}||s_y + \tilde{\phi}_y||_{\mathscr{A}} dy \tag{14}$$

where

$$\bar{a} = P\|s_0\| + P^*\|s_1\| + \frac{(\bar{a}_F)P(1 + P_f T)T^N}{\Gamma(1+n)} + \frac{nP\|G\|_{\infty}}{\Gamma(1+n)}\left(\frac{m-1}{mn-1}\right)^{\frac{m-1}{m}} T^{n-\frac{1}{m}}\|x\|\mathscr{L}m(\tau,S)$$

Suppose $K_T^* = max\{K^*(\xi) : \xi \in \tau\}$ and $P_T = max\{\bar{P}(\xi) : \xi \in \tau\}$. Then

$$\|s_y + \tilde{\phi}_y\|_{\mathscr{A}} \leqslant \|s_y\|_{\mathscr{A}} + \|\tilde{\phi}_y\|_{\mathscr{A}}$$
$$\leqslant K^*(\xi)sup\{\|s(y)\| : 0 \leqslant y \leqslant \xi\} + P(\xi)\|s_0\|_{\mathscr{A}} + K^*(\zeta)sup\{\|\tilde{\phi}_y\| : 0 \leqslant y \leqslant \xi\}$$
$$+ P(\xi)\|\tilde{\phi}_0\|_{\mathscr{A}} \leqslant K_T^* sup\{\|s(y)\| : 0 \leqslant y \leqslant \xi\} + K_T^* P\|\phi(0)\| + P_T\|\phi_0\|_{\mathscr{A}}$$

Set

$$z(\xi) = K_T^* sup\{\|s(y)\| : 0 \leqslant y \leqslant \xi\} + K_T^* P\|\phi(0)\| + P_T\|\phi_0\|_{\mathscr{A}}$$

then

$$\|s_y + \tilde{\phi}_y\|_{\mathscr{A}} \leqslant z(\xi)$$

which implies that 14 can be written as

$$\|s(\xi)\| \leqslant \bar{a} + \frac{(\bar{a}_F)nP(1+P_fT)}{\Gamma(1+n)} \int_0^{\xi} (\xi - y)^{n-1} z(y) dy \tag{15}$$

Note that from 15 and definition of z, we have

$$z(\xi) \leqslant K_T^* P\|phi(0)\| + P_T\|\phi\|_{\mathscr{A}} + K_T^*\bar{a} + \frac{K_T^*\bar{a}_F nP(1+P_fT)}{\Gamma(1+n)} \int_0^{\xi} (\xi - y)^{n-1} z(y) dy$$

By using lemma 5, there exists constant $\hat{P} > 0$ in such a way

$$z(\xi) \leqslant \hat{P}(K_T^* P\|phi(0)\| + P_T\|\phi\|_{\mathscr{A}} + K_T^*\bar{a}) := \tilde{P}, t \in \tau$$

Then we have

$$\|s(\xi)\| \leqslant \bar{a} + \frac{(\bar{a}_F)nP(1+P_fT)}{\Gamma(1+n)} \int_0^{\xi} (\xi - y)^{n-1} \tilde{P}) dy$$

which satisfies that

$$\|s(\xi)\| \leqslant \bar{a} + \frac{(\bar{a}_F)P(1+P_f)T^n}{\Gamma(1+n)} \tilde{P} := P^*$$

As a result, for $\xi \in \tau$,

$$\|u(\xi)\| \leqslant \|s(\xi)\| + \|\S_n(\xi)\phi(0)\| \leqslant P^* + P\|\phi(0)\| := \rho$$

This completes the proof. ■

Theorem 1. Let (H1), (H2), (H3) and (H4) are satisfied, $\phi(0), \phi'(0) \in \varsigma$. Then for each $x \in X_{a^0 d^0}$ and for some m such that $mn > 1$, system 1 has a solution $[-\infty, 0]$ according to x and the mild solution is unique.

Proof. Let $\mathscr{A}C|_{T_1} = \{u : [-\infty, 0] \to \varsigma, u[-\infty, 0] \in \mathscr{A}$ and $u|_{[0,T_1]} \in C([0, T_1], \varsigma)\}$ and

$$\mathscr{B}(1, T_1) = \left\{u \in \mathscr{A}C|_{T_1} max_{y \in [0,T_1]} \|h(y) - u_0 - yu_1\| \leqslant 1 \text{ for } \infty < y \leqslant 0\right\}$$

Then $\mathscr{B}(1,T_1)\mathscr{A}C|_{T_1}$ is closed convex subset of $\mathscr{A}C|_{T_1}$. By using (H1)(i) and (H1)(ii), it is not difficult to see that $F(t,h_t,\int_0^Y f(y,s,u_s)ds)$ is a measurable function on $[0,T_1]$. Let $h \in \mathscr{B}(1,T_1)$, there exists a fixed $\rho^* = \{|u(0)|+1+|u|_{\mathscr{A}}\} > 0$ such that $|h|_{\mathscr{A}C|_{T_1}} \leqslant \rho^*$ using (H1)(iii) and (H2)(iii), for $t \in [0,T_1]$.

$$\begin{aligned} |F(y,h_y,\int_0^Y f(y,s,h_s)ds)| &\leqslant a^0(1+|h_y|_{\mathscr{A}}+P-fT(1+|x|_{\mathscr{A}})) \\ &\leqslant a_F^0(1+\rho^*+P_fT(1+\rho^*)) = \mathscr{K}_F \end{aligned}$$

By Lemma 5(i), Holder inequality and (H1)(iii) and (H2)(iii), we obtain

$$\int_0^\xi (\xi-y)^{n-1}|M_n(\xi-y)F(y,h_y,\int_0^Y f(y,s,h_s)ds)|dy \leqslant \frac{P\mathscr{K}_F}{\Gamma(1+2n)}{T_1}^{2n}$$

Thus, $|(\xi-y)^{n-1}M_n(\xi-y)F(y,h_y,\int_0^Y f(y,s,h_s)ds)|dy$ is Lebesgue integrable with respect to $y \in [0,\xi]$ for all $\xi \in [0,T_1]$ by "Bochner's theorem. Otherwise, in view of Lemma 5(ii) and Holder inequality.

$$\begin{aligned} &\int_0^\xi (\xi-y)^{n-1}|M_n(\xi-y)G(y)x(y)|dy \\ &\leqslant \frac{||G||_\infty}{P}\Gamma(2n)\int_0^\xi (\xi-y)^{2n-1}|x(y)|dy \\ &\leqslant \frac{||G||_\infty P}{\Gamma(2n)}\left(\int_0^\xi (\zeta-y)^{\frac{m}{m-1}(2n-1)}dy\right)^{\frac{m-1}{m}}\left(\int_0^\xi |x(y)|_N^m dy\right)^{\frac{1}{m}} \\ &\leqslant \frac{||G||_\infty P}{\Gamma(2n)}\left(\frac{m-1}{2mn-1}\right)^{\frac{m-1}{m}} T^{2n-\frac{1}{m}}||x||_{\hat{L}^M(\tau,S)} \end{aligned}$$

Thus $(\xi-y)^{n-1}M_n(\xi-y)G(y)x(y)$ is Bochner integrable with respect to $y \in [0,\xi]$ for all $\xi \in [0,T_1]$. We can define $\mathscr{M} : \mathscr{B}(1,T_1) \to \mathscr{A}C|_{T_1}$ as follows:

$$(\mathscr{M}h)(\zeta) = \S_n(\xi)u_0 + \kappa_n(\xi)u_1 + \int_0^{\xi} (\xi - y)^{2n-1} M_n(\xi - y) F(y,h_y, \int_0^{Y} f(y,s,h_s)ds)dy$$

$$+ \int_0^{\xi} (\xi - y)^{2n-1} M_n(\xi - y) G(y)x(y)dy, 0 < \xi \leqslant T_1$$

By the properties of $\S_n$, κ_n, M_n and (H1),(H2) one can verify that $\mathscr{M}$ is a contraction map on $\mathscr{B}(1,T_1)$ which chosen $T_1 > 0$. For $\xi \in [0,T_1]$, it is not difficult to get the following inequality:

$$\begin{aligned}
&|(\mathscr{M}h)(\xi) - u_0 - \xi u_1| \leqslant |\S_n(\xi)u_0 - u_0| + |\kappa_n(\xi)u_1 - \xi u_1| \\
&+ \int_0^{\xi} (\xi - y)^{n-1} |M_n(\xi - y) F(y,h_y, \int_0^{y} f(y,s,h_s)ds)| dy \\
&+ \int_0^{\zeta} (\xi - y)^{n-1} |M_n(\xi - y) G(y)x(y)| dy \\
&\leqslant |\S_n(\xi)u_0 - u_0| + |\kappa_n(\xi)u_1 - \xi u_1| + \frac{P\mathscr{K}_F}{\Gamma(1+2n)} \xi^{2n} + \frac{\|G\|_\infty P}{\Gamma(2n)} \|x\|_{\hat{L}^M(\tau,S)} \zeta^{2n-\frac{1}{m}}
\end{aligned} \tag{16}$$

Since $\{\S_n(\xi)\}_{\xi \geqslant 0}$ and $\{\kappa_n(\xi)\}_{\xi \geqslant 0}$ are strongly continuous operators in ς, one can select T_1 so small, $\upsilon = \frac{1}{3}$ such that

$$|\S_n(\xi)u_0 - u_0| \leqslant \frac{1}{3} and |\kappa_n(\xi)u_1 - \xi u_1| \leqslant \frac{1}{3} \tag{17}$$

Let

$$T_{11} = min \left\{ \frac{1}{3}, \left(\frac{\Gamma(1+2n)}{3P(\mathscr{K}_F T_1^{\frac{1}{m}} + 2n\|G\|_\infty \|x\|_{\hat{L}^M \tau})^{\frac{m-1}{m}}} \right) \right\}$$

Then for all $t \in T_{11}$, we obtain from (16) and (17) that

$$|(\mathscr{M}h)(\xi) - u_0 - \xi u_1| \leqslant 1$$

Hence

$$\mathscr{B}(\mathscr{B}(1,T_1)) \subseteq \mathscr{B}(1,T_1)$$

Let $h_1,h_2 \in \mathscr{B}(1,T_1)$ and $\|h_1\|_{0,T_1},\|h_2\|_{0,T_1} \leqslant \rho^*$.

For $t \in [0,\tau_{11}]$, using Lemma 5(ii),(H1)(iii) and (H2)(iii).

$$\begin{aligned}
&|(\mathscr{M}h_1)(\xi)-(\mathscr{M}h_2)(\xi)| \\
&\leqslant \int_0^{\xi}(\xi-y)^{n-1}|M_n(\xi-y)(F(y,h_{1y},\int_0^y f(y,s,h_{1s})ds)-F(y,h_{2y}),\int_0^y f(y,s,h_s)ds)|dy \\
&\leqslant \frac{P\hat{L}_F(\rho^*)}{\Gamma(2n)}\int_0^{\xi}(\xi-y)^{2n-1}(\|h_{1y}-h_{2y}\|+\int_0^{\xi}\hat{L}_H(\rho^*)(\|h_{1s}-h_{2s}\|)dsdy
\end{aligned}$$

which implies that

$$|(\mathscr{M}h_1)(\xi)-(\mathscr{M}h_2)(\xi)| \leqslant \frac{P\hat{L}_F(\rho^*)(1+\hat{L}_H(\rho^*)T)}{\Gamma(1+2n)}\xi^2 n\|h_1-h_2\|_{0,T_1}$$

Thus

$$\|\mathscr{M}h_1-\mathscr{M}h_2\|_{o,T_1} \quad \leqslant \quad \frac{P\hat{L}_F(\rho^*)(1+\hat{L}_H(\rho^*))}{\Gamma(1+2n)T}T_1^{2n}\|h_1-h_2\|_{0,T_1}$$

Let $T_{12}=\frac{1}{2}(\frac{\Gamma(1+2n)}{P\hat{L}_F(\rho^*)(1+\hat{L}_H(\rho^*)T)})T_1^{2n}$;$T_1=min\{T_{11},T_{12}\}$,

Then $\mathscr{M}$ is a contraction map on $\mathscr{B}(1,T_1)$. It follows from the contraction mapping principle that $\mathscr{M}$ has a unique fixed point $h \in \mathscr{B}(1,T_1)$, and h is the unique mild solution of system (1) with respect to x on $[0,T_1]$. ■

Remark 1. Assume that ς and S are two separable reflexive Banach spaces. If we replace (H1)(i)-(ii) by the condition that $F:\tau\times\varsigma\to\varsigma$ is Holder continuous with respect to t and for any $\rho>0$, there is constant $\hat{L}_F(\rho)>0$ in such a way

$$|F(\xi,\varpi_1)-F(y,\varpi_2)| \leqslant \hat{L}_F(\rho)(|\xi-y|^{\gamma}+|\varpi_1-\varpi_2|)$$

where $\gamma \in (0,1]$, provided that $|\varpi_1|, |\varpi_2| \leqslant \rho$, condition (H2) by the condition

$$F \in \hat{L}^*(\hat{L}^p(\tau,S), \hat{L}^p(\tau,\varsigma)),$$

and the condition (H3) by the condition

$$M_{a^0 d^0} = \hat{L}^m(\tau,S)$$

one can use the some approach to derive the existence of mild solutions.

4. Continuous Dependence

In this subsection, we show that the mild solution of system (1) is continuous dependence on the initial value in sense control term.

Theorem 2. Let $u^1(0), u^2(0) \in \Pi$ where Π is bounded set. Let

$$\begin{cases} u^1(\xi,u_0^1,u_1^1,x) = \S_n(\xi)u_0^1 + \kappa_n(\xi)u_1^1 + \int_0^{\xi}(\xi-y)^{n-1}M_n(\xi-y) \\ F(y,u_y^1, \int_0^y f(y,s,u_s^1)ds)dy + \int_0^{\Xi}(\xi-y)^{n-1}M_n(\xi-y)G(y)x(y)dy, \; 0 \leqslant \xi \leqslant T \\ u^1(\xi) = \phi^1(\xi), \quad -\infty \leqslant \xi \leqslant 0. \end{cases}$$

and

$$\begin{cases} u^2(\xi,u_0^2,u_0^2,w) = \S_n(\xi)u_0^2 + \kappa_n(\xi)u_1^2 + \int_0^{\xi}(\xi-y)^{n-1}M_n(\xi-y)F(y,u_y^2, \\ \int_0^y f(y,s,u_s^2)ds)dy + \int_0^{\xi}(\xi-y)^{n-1}M_n(\xi-y)G(y)w(y)dy, \; 0 \leqslant \xi \leqslant T \\ u^2(\xi) = \phi^2(\xi), \quad -\infty \leqslant \xi \leqslant 0. \end{cases}$$

Then there exists a constant $\mathscr{C}^* > 0$ such that

$$\begin{cases} |u^1(\zeta,u_0^1,u_1^1,x) - u^2(\xi,u_0^2,u_0^2,w)| \leqslant \mathscr{C}^*(|u_0^1 - u_0^2| + \|u_1^1 - u_1^2\| + ||x-w||_{\hat{L}^m\tau}) \; \xi \in \tau \\ |u^1(\xi) - u^2(\xi)| = |\phi^1(\xi) - \phi^2(\xi)|, \quad \infty \leqslant \xi \leqslant \end{cases}$$

where $\mathscr{C}^* = max\left\{P^*P, P^*, P\frac{P||G||_\infty}{\Gamma(2n)}T^{2n-\frac{1}{m}}\right\} > 0$

Proof. Since $u_0^1, u_0^2 \in \Pi$. Π is set which is bounded in ς, applying lemma 6, positive constant ρ exists there in such a way $\|u_s^1\|_{]-\infty,0]}, \|u_s^2\|_{]-\infty,0]}, |u^1|, |u^2| \leqslant \rho$ For $\xi \in \tau$, by using Holder inequality, lemma 5, (H1)(ii),(H2)(ii).

$$
\begin{aligned}
&|u^1(\xi,u_0^1,u_1^1,x) - u^2(\xi,u_0^2,u_1^2,w)| \leqslant |\S_n(\xi)(u_0^1-u_0^2)| + |\kappa_n(\xi)(u_1^1-u_1^2)| \\
&+ \int_0^\xi (\xi-y)^{n-1}|M_n(\xi-y)(F(y,u_y^1,\int_0^y f(y,s,u_s^1)ds) - F(y,u_y^2,\int_0^y f(y,s,u_s^2)ds))|dy \\
&+ \int_0^\xi (\xi-y)^{n-1}|M_n(\xi-y)(G(y)x(s)-G(y)w(y))|dy \\
&\leqslant P|u_0^1-u_0^2| + P^*|u_1^1-u_1^2| + \frac{\hat{L}_F(\rho)P}{\Gamma(2n)}\int_0^\xi (\xi-y)^{2n-1}\big(|u_y^1-u_y^2| \\
&+ \int_0^y \hat{L}_H(\rho^*)|u_s^1-u_s^2|ds\big)dy + \frac{||G||_\infty P}{\Gamma(2n)}\int_0^\xi (\xi-y)^{2n-1}|x(y)-w(y)|_S dy \\
&\leqslant P|u_0^1-u_0^2| + P^*|u_1^1-u_1^2| + \frac{||G||_\infty}{\Gamma(2n)}\xi^{2n-\frac{1}{2m}}\big(\int_0^\xi |w(y)-x(y)|_S^M\big)^{\frac{1}{m}} \\
&+ \frac{\hat{L}_F(\rho)(1+\hat{L}_H(\rho)T)P}{\Gamma(2n)}\int_0^\xi (\xi-y)^{2n-1}||u_y^1-u_y^2||_{\hat{L}^m(\tau,S)}dy
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant P|u_0^1-u_0^2| + P^*|u_1^1-u_1^2| + \frac{||G||_\infty P}{\Gamma(2n)}T^{2n-\frac{1}{m}}\|w-x\|_{\hat{L}^m(\tau,S)} \\
&+ \frac{\hat{L}_F(\rho)(1+\hat{L}_H(\rho)T)P}{\Gamma(2n)}\int_0^\xi (\xi-y)^{2n-1}||u_y^1-u_y^2||_{\hat{L}^m(\tau,S)}dy
\end{aligned}
$$

using Lemma 1 again, we obtain

$$
|u^1(\xi,u_0^1,u_1^1,x) - u^2(\xi,u_0^2,u_1^2,x)| \leqslant \mathscr{C}^*(|u_0^1-u_0^2| + \|u_1^1-u_1^2\|\mathscr{A} + \|w-x\|_{\hat{L}^m(\tau,S)})
$$

for $t \in \tau$. Note that

$$
|u^1(\xi)-u^2(\xi)| \leqslant |\phi^1(\xi)-\phi^2(\xi)|, \quad for \;\; -r \leqslant t \leqslant 0.
$$

This concludes the evidence. ■

5. Optimal Control

Optimal pair's existence for fractional control system (1) are studied in this section. Initially, the Lagrange problem is being consider:

(M) find control $x^\circ \in X_{a^0 d^0}$ such that

$$\mathscr{J}(x^\circ) \leqslant \mathscr{J}(x) \;\; forall \, x \in X_{a^0 d^0}$$

where

$$\mathscr{J}(x) = \int_0^T \mathscr{L}^*(\xi, u_\xi^x, u^x(\xi), x(\xi)) d\xi.$$

u^x denote system's (1) mild solution corresponding to control $x \in X_{a^0 d^0}$.

For solution's existence for problem (M), consider some assumptions:

(H5) (i) Functional $\mathscr{L}^* : \tau \times \mathscr{A} \times \varsigma \times S \to \mathscr{R} \cup \{\infty\}$ is Borel measurable;

(ii) $\mathscr{L}^*(\xi, u, s, \cdot)$ is convex on S for each $u \in \mathscr{A}$, and for every $\xi \in \tau$;

(iii) There exist constant, $d, e \geqslant 0$, $\hat{j} > 0$, ψ is positive and $\psi \in \hat{L}^1(\tau, \mathscr{R})$ such that

$$\mathscr{L}^*(\xi, u, s, x) \geqslant \psi(\xi) + d||u||_{\mathscr{A}} + e||s|| + \hat{j}||x||_s^m.$$

Theorem 3. In (H4) and Theorem 1 under the suppositions, let G is operator which is strongly continuous . Optimal control problem (M) holds one optimal pair, i.e, control $x^\circ \in X_{a^0 d^0}$ exists there in such a way.

$$\mathscr{J}(x^\circ) = \int_0^T \mathscr{L}^*(\xi, u^\circ{}_\xi \xi, u^\circ(\xi)\xi, x^\circ(\xi)) d\xi \leqslant \mathscr{J}(x) \text{ for } x \in X_{a^0 d^0}$$

Proof. If nothing is to be proven in $inf\{\mathscr{J}(x^\circ) : x \in X_{a^0 d^0}\} = +\infty$. Suppose

$$inf\{\mathscr{J}(x) : x \in X_{a^0 d^0}\} = \upsilon < +\infty$$

Using the suppositions (H4), we have $\upsilon > -\infty$.

$$\{(u^p, x^p)\} \subset E_{a^0 d^0} := \{(u, x); u \text{ is system's (1) mild solution corresponding to } x \in X_{a^0 d^0}\}.$$

in such a way $\mathscr{J}(u^p,x^p) \to \upsilon$ as $p \to +\infty$. As $\{x^p\} \subseteq X_{a^0d^0}$, $p = 1,2,...,\{x^p\}$ is subset of the separable reflexive Banach space $\hat{L}^m(\tau,S)$ which is bounded, a subsequence (we denote it by $\{x^p\}$) exists there and $x^\circ \in \hat{L}^m(\tau,S)$ such that

$$x^p \to x^\circ in \hat{L}^m(\tau,S).$$

Since $X_{a^0d^0}$ is closed and convex, which leads to Marzur Lemma $x^\circ \in X_{a^0d0}$.

Assume that $u^p \in \mathscr{AC}$ denote solution's corresponding sequence of the integral equation.

$$u^p(\xi) = \S_n(\xi)u_0 + \kappa_n(\xi)u_1 + \int_0^\xi (\xi-y)^{n-1}M_n(\xi-y)F(y,u_y^p,\int_0^y f(y,s,u_(^p s))ds)dy$$
$$+\int_0^\xi (\xi-y)^{n-1}M_n(\xi-y)G(y)x^p(y)dy,\xi \in \tau$$

In view of Lemma 1 and 6, it can be verified that there is $\rho > 0$ in such a way

$$||u^p||_{\mathscr{AC}} \leqslant \rho,\, where\ p = 0,1,2,...$$

here u° represent solution according to x°,

$$u^\circ(\xi) = \S_n(\xi)u_0 + \kappa_n(\xi)u_1 + \int_0^\xi (\xi-y)^{n-1}M_n(\xi-y)F(y,u_y^0,\int_0^y f(y,s,u_s^0)ds)dy$$
$$+\int_0^\xi (\xi-y)^{n-1}M_n(\xi-y)G(y)x^\circ(y)dy,\ \xi \in \tau$$

Hence, for $\xi \in \tau$ by condition (H1)(ii), Lemma 5(i) and Holder inequality, we have the following inequality:

$$|u^p(\xi)-u^\circ(\xi)| \leqslant \int_0^\xi |(\xi-y)^{n-1}M_n(\xi-y)[F(y,u_y^p,\int_0^y f(y,s,u_s^p)-$$
$$F(y,u_y^0,\int_0^y f(y,s,u_s^0)ds)]dy + \int_0^\xi (\xi-y)^{n-1}|M_n(\xi-y)|x^p(y)G(y)-x^\circ(y)G(y))|dy$$

$$\leqslant \frac{\hat{L}_F(\rho)P}{\Gamma(2n)} \int_0^{\xi} (\xi - y)^{2n-1} \big(|u_y^1 - u_y^0| + \int 0y\hat{L}_H(\rho)|u_s^1 - u_s^0|ds\big)|dy$$
$$+ \frac{P}{\Gamma(2n)} \int_0^{\zeta} (\xi - y)^{2n-1} |x^p(y)G(y) - x^{\circ}(y)G(y)|dy$$
$$\leqslant \frac{\hat{L}_F(\rho)(1 + \hat{L}_H(\rho))P}{\Gamma(2n)} \int_0^{\xi} (\xi - y)^{2n-1} ||u_y^p - u_y^{\circ}||dy$$
$$+ \frac{P}{\Gamma(2n)} T^{2n-\frac{1}{m}} \big(\int_0^T |G(y)x^p(y) - G(y)x^{\circ}(y)|^M dy\big)^{\frac{1}{m}}$$
$$= \eta_p^{(1)} + \eta_p^{(2)}$$

Since G is strongly continuous, $\|Gx^p - Gx^{\circ}\| \to 0$ as $p \to \infty$, by applying Lemma 3 we have

$$\int_0^T \big|G(y)x^p(y) - G(y)x^{\circ}(y)\big|^m dy \to 0, \quad \text{as } p \to \infty,$$

which implies that $\eta_p^{(2)} \to 0$ as $p \to \infty$. Moreover, we have

$$\big|u_{\zeta}^p - u_{\xi}^{\circ}\big| \leqslant |\eta_p^{(2)}| + \frac{L_F(\rho)(1 + \hat{L}_H(\rho)T)P|}{\Gamma(2n)} \int_0^{\xi} (\xi - y)^{2n-1} \big|u_y^p - u_y^{\circ}\big|dy.$$

By virtue of Gronwall inequality again, there exists a $P^* > 0$ such that

$$\big|u_{\xi}^p - u_{\xi}^{\circ}\big| \leqslant P^*|\eta_p^{(2)}|,$$

which yields that

$$u^p \to u^{\circ} \;\; in \;\; \mathscr{AC} \;\; as \;\;\; p \to \infty.$$

Balder assumption is provided by assumption (H5) . It can be concluded by using Balder's theorem.

$$(u,x) \to \int_0^T \mathscr{L}^*(\xi, u_{\xi}, w_{\xi}, x(\xi))d\xi$$

in the weak topology of $L^m(\tau,S) \subset L^1(\tau,S)$ is lower semi-continuous, and the strong topology of $L^1(\tau,\varsigma)$. τ is weakly lower semi-continuous on $L^m(\tau,S)$, by using (H5)(iv)

$\tau > -\infty$, τ attains its infimum at $x\circ \in X_{a^\circ d^\circ}$, i.e,

$$\begin{aligned} \upsilon &= \lim_{p\to\infty} \int_0^T \mathscr{L}^*(\xi, u^p_\xi, u^p(\xi), x^p(\xi))d\xi \\ &\geqslant \int_0^T \mathscr{L}^*(\xi, u^\circ_\xi, u^\circ(\xi), x^\circ(\xi))d\xi \\ &= \mathscr{J}(x^\circ) \\ &\geqslant \upsilon. \end{aligned}$$

The proof is completed. ■

Finally, we present an example To illustrate the main results, finally some examples are presented.

Example 1. Suppose the problem:

$$\begin{cases} {}^c_0D^\delta_\xi u(\xi,v) - \dfrac{\partial^2}{\partial v^2}u(\xi,v) = M(\xi, \int_{-\infty}^\xi M_1(s-\xi)u(s,v)ds, \int_0^\xi M_2(s,v,\rho,\rho-s) \\ \qquad u(\rho,v)d\rho ds) + \int_{[0,1]} \Psi(v,s)x(s,\xi)ds, q \in (\dfrac{6}{5},1)\ \zeta \in \tau \\ u(\xi,0) = u(\xi,1) = 0, \xi > 0 \\ u(\xi,v) = phi(\xi,v),\ u'(\xi,v) = \phi'(\xi,v), -\infty \leqslant t \leqslant 0, v \in [0,1]. \end{cases} \tag{18}$$

here ϕ is continuous and satisfies some conditions,$u \in L^2(\tau \times [0,1])$,and $\Psi : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Furthermore, we make some hypothesis:

(h1) In $]-\infty,0]$, $M_1(s)$ is continuous and $\int_{-\infty}^0 M_1^2(s)ds < \infty$.

(h2) In $\tau \times [0,1]$, M is continuous and there exists a positive constant μ such that

$$|M(\zeta,y_1\theta_1) - M(\xi,y_2,\theta_2)| \leqslant \mu(\gamma|y_1-y_2| + |\theta_1-\theta_2|).\text{for all } \xi \in \tau$$

where $\gamma = \left(\frac{-1}{2\varpi}\int_{-\infty}^0 M_1^2(s)ds\right)^{\frac{-1}{2}}$.

(h2') M is continuous on $\tau \times [0,1] \times [0,1]$ and there exist a constant $\nu > 0$ such that

$$|M(\xi,y,\theta)| \leqslant \nu(\gamma|y| + |w|), \text{ for all } t \in \tau$$

(h3) $M_2(\xi,v,s) \geqslant 0$ is continuous in $\tau \times [0,1] \times] - \infty,0]$ and $\int_{-\infty}^{0} M_2(\xi,v,s) = \psi(\xi,v) < \infty$ and $\nu = max\psi(\xi,v) : \xi \in \tau, v \in [0,1]$.

Let $\varsigma = S = L^2(0,1)$ equipped with the usual norm $\| \cdot \|_L^2$,and $D(E) = \Theta^{2,2}(0,1) \cap \Theta_0^{1,2}(0,1)$, and $Eu = -\frac{\partial^2 u}{\partial v^2}$ for $u \in D(E)$. Then E can create a strong continuous cosine family $\mathscr{C}(t)_{t \geqslant 0}$ in ς. Function of controls are $x : \mathscr{C}(u[0,1]) \to \mathbb{R})$, such that $x \in L^2(\mathscr{C}(u[0,1]))$. This claim is that $\xi \to x(\cdot,\xi)$ going from τ into S is measurable. Let $X(\xi) = x \in S : \|x\|_S \leqslant \varphi$, where $\varphi \in L^2(\tau,\mathbb{R}^+)$. we set the admissible control $X_{a^o d^o}$ to all $x \in L^2(\mathscr{C}(u[0,1]))$ such that $\|x(\cdot,\xi)\|_{L^2 \mathscr{C}(u[0,1])} \leqslant \varphi(t)$, a.e.

Suppose $\omega \leqslant 0$, define the phase space

$$\mathscr{A} = \left\{ \varpi \in C(] - \infty,0],\varsigma) : lim_{y \to -\infty} e^{\omega y} \varpi(y) \text{ exists } in \varsigma \right\}$$

and also suppose that

$$\|\varpi\|_{\mathscr{A}} = sup_{-\infty < y \leqslant 0} \{ e^{-\omega y} \|\varpi(y)\| \}$$

Then $(\mathscr{A}, \| \cdot \|_{\mathscr{A}})$ is a Banach space which satisfies (H1)-(H3) with $K = 1, H(\zeta) = max\{1, e^{\omega \xi}\}, \bar{P}(\xi) = e^{-\omega \xi}$

For $(\xi,\varpi) \in [0,1] \times \mathscr{A}$, where $\varpi(y)(s) = \phi(y,s), (y,s) \in] - \infty,0] \times [0,1]$, let

$$u(\xi)(s) = u(\xi,s),$$

$$f(\xi,\varpi)(s) = \int_{-\infty}^{0} M_2(\xi,s,y)\varpi(y)(s)dy,$$

$$F\left(\xi,\varpi, \int_0^{\xi} f(y,\varpi)dy\right)(s) = M\left(\xi, \int_{-\infty}^{0} M_1(y)\varpi(y)(s)dy, \int_0^{\xi} f(y,\varpi)(s)dy\right),$$

$$G(\xi)x(\zeta)(s) = \int_{[0,1]} \Psi(s,y)x(y,\xi)dy,$$

Then the system (1) can be abstracted as the problem (18).

Consider cost function:

$$\mathscr{M}(x) = \int_0^T \mathscr{J}(\zeta, u^x_\xi, u^x(\xi), x(\xi))d\xi,$$

where

$$\mathscr{J} : \tau \times C^{1,0}(]-\infty,0] \times [0,1] \times L^2(\tau \times [0,1])) \to \mathbb{R} \cup +\infty$$

for $u \in C^{1,0}(]-\infty,T] \times [0,1])$ and $x \in L^2([0,1] \times \tau)$,

$$\begin{aligned}\mathscr{J}(\xi, u^x_\xi, u^x(\xi), x(\xi))(s) &= \int_{[0,1]} \int_{-\infty}^0 |u^x(\xi + y, s)|^2 dyds + \int_{[0,1]} |u^x(\xi, s)|^2 ds \\ &\quad + \int_{[0,1]} |x(s,\xi)|^2 ds.\end{aligned}$$

It can be easily verified that all assumptions in Theorem 3 are satisfied. These results can be applied to problem (18).

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