Available Online
International Journal of Advancements in Mathematics
http://www.scienceimpactpub.com/IJAM

New fractional inequalities for Convexities and pre-invexities in aspects of Atangana-Baleanu fractional operators

Sabila Ali, Rana Safdar Ali [11], *Wajid Iqbal*[1], *Naila Talib*[1], *Humira saif*[1], *Saima Batool*[2]

[1] Government Women College Joharabad, Punjab, Pakistan

[2] Department of Mathematics and Statistics, The University of Lahore, Sargodha 40100, Pakistan

[3] University of Agriculture Faisalabad, Faisalabad, Pakistan. Email addresses:
sabila21imran@gmail.com (S. Ali)
safdar.ali@math.uol.edu.pk (R. S. Ali)
wajid.iqbal@math.uol.edu.pk (W. Iqbal)
20nailatalib@gmail.com (N. Talib)
humirasaifuol@gmail.com (H. Saif)
saimabatool1311@gmail.com (S. Batool)

Abstract

This paper presents the innovative idea to obtain the improve version of well known inequalities for different type of convexities and pre-invexities by Atangana-Baleanu (AB) fractional operators. We establish the fractional inequalities for h-Godunova Levin function, defined by Ohud Almutairi and Adem Kiliçman with the AB-operators. Moreover, we discuss the significant behavior of Hermite-Hadamard type fractional integral inequalities to the AB-fractional operator and discuss its applications.

Keywords: Fractional Inequalities, h-Godunova-Levin convex and preinvex function, Hermite-Hadamard inequality, Fractional operators.

1 Introduction

The convex function has widely utilized function by the researchers to make fruitful innovations in literature and in real world problems. The concept of convexities have prevailed many mathematical problems and achieved sustainable goals, due to this most of the researchers are frequently introduced the new functions related to convex. Convex functions and its generalizations have obtained immense applications in the field of fractional-inequality theory due to wide range of features, and usefulness for numerous work such as numerical integration, convex programming and special means. To markable theory of convex function and its significant applications have been discussed [24, 25, 26, 27, 1, 28, 29, 30, 31, 32].

[1]Corresponding author

Definition 1. *[7, 8] The convex function $\wp : J \to \mathbb{R}$, $J \subset \mathbb{R}$ is defined for $t \in [0,1]$, $\forall m_1, m_2 \in J$ as follows*

$$\wp\Big[tm_1 + (1-t)m_2\Big] \leq t\wp(m_1) + (1-t)\wp(m_2).$$

Definition 2. *[6] The pre-invex function $\wp : J \to \mathbb{R}$, $J \subset \mathbb{R}$ is defined for $m_1, m_2 \in J$ and $\lambda \in [0,1]$ as follows*

$$\wp(m_2 + \lambda\zeta(m_1, m_2)) \leq \lambda\wp(m_1) + (1-\lambda)\wp(m_2),$$

where J is an invex set with respect to ζ.

Definition 3. *[33] Let $h : (0,1) \to \mathbb{R}$. A non-negative function $\wp : J \to \mathbb{R}$ is said to be h-Godunova-Levin, $\forall m_1, m_2 \in J$ and $t \in (0,1)$, if the following inequality holds:*

$$\wp(tm_1 + (1-t)m_2) \leq \frac{\wp(m_1)}{h(t)} + \frac{\wp(m_2)}{h(1-t)}$$

Definition 4. *[33] Let $h : (0,1) \to \mathbb{R}$. A function $\wp : J \to \mathbb{R}$ is said to be h-Godunova-Levin pre-invex with respect to ζ, if the following inequalities holds*

$$\wp(m_1 + t\zeta(m_2, m_1)) \leq \frac{\wp(m_1)}{h(1-t)} + \frac{\wp(m_2)}{h(t)}$$

where $\forall m_1, m_2 \in J$, $t \in [0,1]$ and J be an invex set.

Fractional calculus has rapidly developed in applied mathematics and analysis of inequalities. Fractional operators resolved many problems related to the extensions and generalizations of well known inequalities by successfully implemented the fractional operators having modified version of special functions as its kernel for different type of convexities and pre-invexities [24, 25, 26, 27, 1, 28, 29, 30, 31, 32]. The gradually development of fractional operators have increased the demand of special functions, which utilized act as its kernel and discussed many applications by the researchers [18, 19, 20]. Atangana-Baleanu fractional integral operator which has revealed the researcher's attention towards this tool because of its efficiency and effectiveness in applying to engineering and many other fields as it has more powerful properties than the previously known operators, is defined as:

Definition 5. *[21, 22] Consider $\wp \in H^1[u,v]$. The Atangana-Baleanu integral operators $I_x^\alpha\{\wp(x)\}$ and $I_x^\alpha\{\wp(x)\}$ with $u > v$ and $\alpha \in [0,1]$ are defined as:*

$${}_{u}^{AB}I_x^\alpha\{\wp(x)\} = \frac{1-\alpha}{B(\alpha)}\wp(x) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}\wp(t)dt,$$

and

$${}^{AB}I_v^\alpha\{\wp(x)\} = \frac{1-\alpha}{B(\alpha)}\wp(x) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_x^v (t-x)^{\alpha-1}\wp(t)dt,$$

where $B(\alpha)$ is normalization function with $B(0) = B(1) = 1$ and the Gamma function $\Gamma(\alpha)$ is defined in the next definition.

Definition 6. *[23]The integral representation of gamma function is defined as*

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx$$

for, $\Re(\alpha) > 0$.

Fractional integral inequalities is one of the emerging branch of fractional calculus[12, 13, 14, 15, 16, 17]. Hermite-Hadamard and its related type inequalities are highly worked by the researchers as these are proved to be helpful tools in the field of analysis, numerical integration error estimations and many others applied sciences, is defined as:

$$\wp\Big(\frac{m_1+m_2}{2}\Big) \leq \frac{1}{m_2-m_1}\int_{m_1}^{m_2}\wp(x)dx \leq \frac{\wp(m_1)+\wp(m_2)}{2}$$

for convex function [2, 8, 9, 10, 3, 11] $\wp : J \to R, m_1, m_2 \in J, m_1 < m_2, m_1, m_2 \in R, J \subseteq R$ The fractional version of Hermite-Hadamard inequality using well known Riemann-Lioville fractional integral operator is defined as:

Definition 7. *[37]*

$$\wp\left(\frac{m_1+m_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(m_2-m_1)^{\alpha}}\Big[J^{\alpha}_{m_1^+}\wp(m_2) + J^{\alpha}_{m_2^-}\wp(m_1)\Big] \leq \frac{\wp(m_1)+\wp(m_2)}{2}.$$

The purpose of our work is to develop new inequalities using AB-fractional integral operator to give a new approach to the inequalities theory.

2 Main Results

In this section, we develop new version of fractional inequalities of h-Godunova Levin convex function using Atangana-Baleanu fractional operator, and also modification of Hermite-Hadamard type Inequalities by Atangana-Baleanu fractional operators (ABFO) has been discussed.

Lemma 1. *[4] Let $\theta \in [0,1]$, there exist two cases*

1. *For $n \in [0,1]$, then we have*
$$(1-\theta)^n \leq 2^{1-n} - \theta^n.$$
2. *For $n \in [1,\infty]$, then we have*
$$(1-\theta)^n \geq 2^{1-n} - \theta^n.$$

Lemma 2. *Let $\wp : [m_1, m_2] \to \mathbb{R}$ be a twice differentiable mapping on (m_1, m_2) with $m_1 < m_2$.If $\wp'' \in L[m_1, m_2]$, then the following fractional integral inequality holds for Atangana-Baleanu integral operator defined as in definition 5:*

$$\frac{\wp(m_1)+\wp(m_2)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{(m_2-m_1)^{\alpha}}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^{\alpha}}$$
$$\left[{}^{AB}_{m_1}I^{\alpha}_{m_2}\{\wp(m_2)\} + {}^{A\,B}_{m_2}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]$$
$$= \frac{(m_2-m_1)^2}{2}\int_0^1 \frac{1-\theta^{\alpha+1}-(1-\theta)^{\alpha+1}}{\alpha+1}\wp''(\theta m_1 + (1-\theta)m_2)d\theta.$$

Proof. Consider the following integral

$$\int_0^1 (1-\theta^{\alpha+1}-(1-\theta)^{\alpha+1})\wp''(\theta m_1+(1-\theta)m_2)d\theta = \int_0^1 \wp''(\theta m_1+(1-\theta)m_2)d\theta$$
$$-\int_0^1 \theta^{\alpha+1}\wp''(\theta m_1+(1-\theta)m_2)d\theta - \int_0^1 (1-\theta)^{\alpha+1}\wp''(\theta m_1+(1-\theta)m_2)d\theta$$

Integrating two times by parts, gives

$$\frac{(m_2-m_1)^2}{2(\alpha+1)}\int_0^1 (1-\theta^{\alpha+1}-(1-\theta)^{\alpha+1})\wp''(\theta m_1+(1-\theta)m_2)d\theta = \frac{\wp(m_1)+\wp(m_2)}{2}$$
$$-\frac{\alpha}{2}\int_0^1 (1-\theta)^{\alpha-1}\wp(\theta m_1+(1-\theta)m_2)d\theta$$
$$-\frac{\alpha}{2}\int_0^1 \theta^{\alpha-1}\wp(\theta m_1+(1-\theta)m_2)d\theta.$$

Substituting $\theta m_1+(1-\theta)m_2 = x$ and by adding subtracting the terms $\frac{(1-\alpha)\wp(m_1)}{B(\alpha)}$, $\frac{(1-\alpha)\wp(m_2)}{B(\alpha)}$, leads to the result. □

By using the lemma 2, we will present the next result.

Theorem 1. *Let $\wp : [0, m_2] \to \mathbb{R}$ be a differentiable mapping.If $|\wp''|^q$ is measurable and h-Godunova Levin convex on $[0, m_2]$ with $0 < m_1 < m_2$, then for the Atangana-Baleanu fractional operator (ABFO) is defined in 5, then the following inequality holds*

$$\left|\frac{\wp(m_1)+\wp(m_2)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{(m_2-m_1)^\alpha}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[{}^{AB}_{m_1}I^{\alpha}_{m_2}\{\wp(m_2)\} + {}^{A\ B}_{m_2}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\right|$$
$$\leq \frac{(m_2-m_1)^2(2^{1-\alpha}-1)}{2(\alpha+1)}\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}}.$$

Proof. If we consider the absolute value of lemma 2 and using Hölder's integral inequality with lemma 1, we have

$$\begin{aligned}
&\left|\frac{\wp(m_1)+\wp(m_2)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{(m_2-m_1)^\alpha}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[{}^{AB}_{m_1}I^\alpha_{m_2}\{\wp(m_2)\}+{}^{A\ B}_{m_2}I^\alpha_{m_1}\{\wp(m_1)\}\right]\right| \\
&\leq \frac{(m_2-m_1)^2}{2}\int_0^1 \left|\frac{1-\theta^{\alpha+1}-(1-\theta)^{\alpha+1}}{\alpha+1}\right||\wp''(\theta m_1+(1-\theta)m_2)|d\theta. \\
&\leq \frac{(m_2-m_1)^2}{2(\alpha+1)}\left(\int_0^1 |\theta^{\alpha+1}+(1-\theta)^{\alpha+1}-1|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1 |\wp''(\theta m_1+(1-\theta)m_2)|^q d\theta\right)^{\frac{1}{q}}. \\
&\leq \frac{(m_2-m_1)^2}{2(\alpha+1)}\left(\int_0^1 |\theta^{\alpha}+(1-\theta)^{\alpha}-1|^p d\theta\right)^{\frac{1}{p}}\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}}. \\
&\leq \frac{(m_2-m_1)^2}{2(\alpha+1)}\Big(\int_0^1 (2^{1-\alpha}-1)^p d\theta\Big)^{\frac{1}{p}}\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}}. \\
&= \frac{(m_2-m_1)^2}{2(\alpha+1)}\big(2^{1-\alpha}-1\big)\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ □

Theorem 2. *With the assumption of theorem 1 with the power mean inequality, then we have the following result*

$$\begin{aligned}
&\left|\frac{\wp(m_1)+\wp(m_2)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{(m_2-m_1)^\alpha}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^{\ \alpha}}\left[{}^{AB}_{m_1}I^\alpha_{m_2}\{\wp(m_2)\}+{}^{A\ B}_{m_2}I^\alpha_{m_1}\{\wp(m_1)\}\right]\right| \\
&\leq \frac{(m_2-m_1)^2(2^{1-\alpha}-1)^{1-\frac{1}{q}}}{2(\alpha+1)}\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{|\theta^\alpha+(1-\theta)^\alpha-1|}{h(\theta)}d\theta\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Proceeding again as in theorem 1 with Power mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{\wp(m_1)+\wp(m_2)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{(m_2-m_1)^\alpha}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[{}^{AB}_{m_1}I^\alpha_{m_2}\{\wp(m_2)\}+{}^{A\ B}_{m_2}I^\alpha_{m_1}\{\wp(m_1)\}\right]\right| \\
&\leq \frac{(m_2-m_1)^2}{2}\int_0^1 \left|\frac{1-\theta^{\alpha+1}-(1-\theta)^{\alpha+1}}{\alpha+1}\right||\wp''(\theta m_1+(1-\theta)m_2)|d\theta. \\
&\leq \frac{(m_2-m_1)^2}{2(\alpha+1)}\left(\int_0^1 |\theta^{\alpha}+(1-\theta)^{\alpha}-1|d\theta\right)^{1-\frac{1}{q}}\left(\int_0^1 |\theta^{\alpha}+(1-\theta)^{\alpha}-1||\wp''(\theta m_1+(1-\theta)m_2)|^q d\theta\right)^{\frac{1}{q}} \\
&= \frac{(m_2-m_1)^2}{2(\alpha+1)}(2^{1-\alpha}-1)^{1-\frac{1}{q}}\left(|\wp''(m_1)|^q+|\wp''(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1 \frac{|\theta^\alpha+(1-\theta)^\alpha-1|}{h(\theta)}d\theta\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ □

Theorem 3. *Let $\wp:[m_1,m_2]\to\mathbb{R}$ be a h-Godunova Levin convex function where $0<m_1<m_2$ and $\wp\in L_1[m_1,m_2]$,with $h:(0,1)\to\mathbb{R}$ is a positive function and $h(\theta)\neq 0$,then for*

Atangana-Baleanu fractional integral operator defined as in definition 5, we have

$$\frac{h(\frac{1}{2})}{2}\wp\left(\frac{m_1+m_2}{2}\right) \leq \frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[{}^{AB}_{m_2}I^{\alpha}_{m_1}\{\wp(m_1)\} + {}^{A\ B}_{m_1}I^{\alpha}_{m_2}\{\wp(m_2)\}\right]$$
$$-\frac{(1-\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[\wp(m_1)+\wp(m_2)\right] \leq \frac{\wp(m_1)+\wp(m_2)}{2}\alpha\int_0^1\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]\theta^{\alpha-1}d\theta.$$

Proof. Since $\wp$ is h-Godunova-Levin convex on the interval $[m_1, m_2]$, let $x, y \in [m_1, m_2]$ and $\zeta \in (0, 1)$, we have

$$\wp(\zeta x + (1-\zeta)y) \leq \frac{\wp(x)}{h(\zeta)} + \frac{\wp(y)}{h(1-\zeta)},$$

where by taking

$$x = \theta m_1 + (1-\theta)m_2, y = (1-\theta)m_1 + \theta m_2$$

and

$$\zeta = \frac{1}{2}$$

leads to

$$\wp(\frac{m_1+m_2}{2}) \leq \frac{1}{h(\frac{1}{2})}\left[\wp(\theta m_1 + (1-\theta)m_2) + \wp((1-\theta)m_1 + \theta m_2)\right].$$

Multiplying the inequality by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\theta^{\alpha-1}$ and integrating the resulting inequality on the interval $[0, 1]$ with respect to θ and by using some simple calculus, gives

$$h(\frac{1}{2})\wp(\frac{m_1+m_2}{2})\frac{1}{B(\alpha)\Gamma(\alpha)} \leq \frac{1}{(m_2-m_1)^\alpha}\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$$
$$\left[\int_{m_1}^{m_2}(m_2-w)^{\alpha-1}\wp(w)dw + \int_{m_1}^{m_2}(t-m_1)^{\alpha-1}\wp(t)dt.\right]$$

or

$$\frac{h(\frac{1}{2})}{2}\wp(\frac{m_1+m_2}{2}) \leq \frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[{}^{AB}_{m_1}I^{\alpha}_{m_2}\{\wp(m_2)\} + {}^{A\ B}_{m_2}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]$$
$$-\frac{(1-\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\left[\wp(m_1)+\wp(m_2)\right] \tag{1}$$

For right side of inequality, again using h-Godunova-Levin convexity of $\wp$, we have

$$\wp(\theta m_1 + (1-\theta)m_2) \leq \frac{\wp(m_1)}{h(\theta)} + \frac{\wp(m_2)}{h(1-\theta)}$$

and

$$\wp((1-\theta)m_1 + \theta m_2) \leq \frac{\wp(m_1)}{h(1-\theta)} + \frac{\wp(m_2)}{h(\theta)}.$$

Addition of these two inequalities , gives

$$\wp(\theta m_1 + (1-\theta)m_2) + \wp((1-\theta)m_1 + \theta m_2) \leq (\wp(m_1)+\wp(m_2))\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right].$$

Proceeding as above, we reach

$$\frac{B(\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\Big[{}^{AB}_{m_1}I^{\alpha}_{m_2}\{\wp(m_2)\}+{}^{A\,B}_{m_2}I^{\alpha}_{m_1}\{\wp(m_1)\}\Big]-\frac{(1-\alpha)\Gamma(\alpha)}{2(m_2-m_1)^\alpha}\big[\wp(m_1)+\wp(m_2)\big]$$
$$\leq[\wp(m_1)+\wp(m_2)]\int_0^1\left[\frac{1}{h(\theta)}+\frac{1}{h(1-\theta)}\right]\theta^{\alpha-1}d\theta. \tag{2}$$

Combining (1)and(2), we reach to the required inequality. □

Corollary 1. *Replacing $h(\theta)$ by $\frac{1}{h(\theta)}$ and $\alpha=1$ in Theorem 3, we obtain Hermite-Hadamard type inequality for h-convex function by M. Z. Sarikaya et. al [5].*

$$\frac{1}{2h(\frac{1}{2})}\wp\left(\frac{m_1+m_2}{2}\right)\leq\frac{1}{(m_2-m_1)}\int_{m_1}^{m_2}\wp(\theta)d\theta\leq[\wp(m_1)+\wp(m_2)]\int_0^1 h(\theta)d\theta..$$

Corollary 2. *Choosing $\alpha=1$ and $h(\theta)=\frac{1}{\theta^s}$, we obtain theorem(2.1) by Dragomir in [**?**] .*

$$2^{s-1}\wp\left(\frac{m_1+m_2}{2}\right)\leq\frac{1}{m_2-m_1}\int_{m_1}^{m_2}\wp(\theta)d\theta\leq\frac{\wp(m_1)+\wp(m_2)}{s+1}.$$

Lemma 3. *Consider a function $\wp: J=[m_1,m_1+\zeta(m_2,m_1)]\to\mathbb{R}$ with $m_1,m_2\in\mathbb{R}$, $\wp\in L_1[m_1,m_1+\zeta(m_2,m_1)]$ be a differentiable function where $J=[m_1,m_1+\zeta(m_2,m_1)]$ is taken to be an open invex set with respect to $\zeta: J\times J\to\mathbb{R}$ with $\zeta(m_2,m_1)>0$ for $m_1,m_2\in J$. Then for Atangana-Baleanu fractional integral operators defined as in definition 5 , the following inequality holds with $n=m_1+\zeta(m_2,m_1)$*

$$\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^\alpha(m_2,m_1)}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2\zeta^\alpha(m_2,m_1)}$$
$$\left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\}+{}^{AB}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]$$
$$=\frac{\zeta(m_2,m_1)}{2}\int_0^1[\theta^\alpha-(1-\theta)^\alpha]\wp'(m_1+\theta\zeta(m_2,m_1))d\theta.$$

Proof. If we consider the following fractional integral

$$I=\int_0^1\theta^\alpha\wp'(m_1+\theta\zeta(m_2,m_1))d\theta+\int_0^1-(1-\theta)^\alpha\wp'(m_1+\theta\zeta(m_2,m_1))d\theta,$$

and

$$I=I_1+I_2.$$

taking integrating by parts of I_1, we have

$$I_1=\Big[\frac{\wp(n)}{\zeta(m_2,m_1)}-\frac{\alpha}{\zeta(m_2,m_1)}\int_{m_1}^{n}\big(\frac{x-m_1}{\zeta(m_2,m_1)}\big)^{\alpha-1}\frac{\wp(x)}{\zeta(m_2,m_1)}dx\Big]$$
$$I_1=\frac{\wp(n)}{\zeta(m_2,m_1)}-\frac{B(\alpha)\Gamma(\alpha)}{\zeta^{\alpha+1}(m_2,m_1)}{}^{AB}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha+1}(m_2,m_1)}\wp(m_1)$$

On the same manner as above, we have

$$I_2 = \frac{\wp(m_1)}{\zeta(m_2,m_1)} - \frac{B(\alpha)\Gamma(\alpha)}{\zeta^{\alpha+1}(m_2,m_1)} {}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\} + \frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha+1}(m_2,m_1)}\wp(n).$$

Multiplying I by $\frac{\zeta(m_2,m_1)}{2}$, We get the result. □

By using lemma 3, we present the following theorem.

Theorem 4. *If we consider a function $\wp : J = [m_1, m_1 + \zeta(m_2,m_1)] \to (0,\infty)$ with $J \in \mathbb{R}$, be a differentiable function on J. Also, suppose that $|\wp'|$ is a h-Godunova-Levin preinvex function on J, taking $n = m_1 + \zeta(m_2,m_1)$, then inequality holds for (ABFO) as follows*

$$\Bigg|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha}(m_2,m_1)}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2\zeta^{\alpha}(m_2,m_1)} \left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\} + {}^{A\,B}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\Bigg| \leq \frac{\zeta(m_2,m_1)}{2}[|\wp'(m_1)|+|\wp'(m_2)|]\int_0^1 \frac{|\theta^{\alpha}-(1-\theta)^{\alpha}|}{h(\theta)}d\theta.$$

Proof.

$$\begin{aligned}
&\Bigg|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha}(m_2,m_1)}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2\zeta^{\alpha}(m_2,m_1)} \left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\} + {}^{A\,B}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\Bigg| \\
&\leq \frac{\zeta(m_2,m_1)}{2}\int_0^1 |\theta^{\alpha}-(1-\theta)^{\alpha}||\wp'(m_1+\theta\zeta(m_2,m_1))|d\theta. \\
&\leq \frac{\zeta(m_2,m_1)}{2}\int_0^1 |\theta^{\alpha}-(1-\theta)^{\alpha}|\left|\frac{\wp'(m_1)}{h(\theta)}+\frac{\wp'(m_2)}{h(1-\theta)}\right|d\theta \\
&\leq \frac{\zeta(m_2,m_1)}{2}\int_0^1 |\theta^{\alpha}-(1-\theta)^{\alpha}|\left[\frac{|\wp'(m_1)|}{h(\theta)}+\frac{\wp'(m_2)}{h(1-\theta)}\right]d\theta \\
&= \frac{\zeta(m_2,m_1)}{2}[|\wp'(m_1)|+|\wp'(m_2)|]\int_0^1 \frac{|\theta^{\alpha}-(1-\theta)^{\alpha}|}{h(\theta)}d\theta
\end{aligned}$$

which are the required inequality. □

Theorem 5. *Suppose that $\wp : J = [m_1, m_1 + \zeta(m_2,m_1)] \to (0,\infty)$ with $J \in \mathbb{R}$, be a differentiable real valued function on J. Also, suppose that $|\wp'|^q$ is a h-Godunova-Levin preinvex function on J with $p > 1$ and $q = \frac{p}{p-1}$, taking $n = m_1 + \zeta(m_2,m_1)$, then for Atangana-Baleanu fractional integral operators defined in definition 5 , we have*

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha}(m_2,m_1)}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2\zeta^{\alpha}(m_2,m_1)}$$
$$\left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\}+{}^{A}{}^{B}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\Big|$$
$$\leq\frac{\zeta(m_2,m_1)}{2}\left(|\wp'(m_1)|^q+|\wp'(m_2)|^q\right)^{\frac{1}{q}}\left(\int_0^1\left|\theta^{\alpha}-(1-\theta)^{\alpha}\right|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}}.$$

Proof. If we consider the lemma 3, we have

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha}(m_2,m_1)}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2\zeta^{\alpha}(m_2,m_1)}$$
$$\left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\}+{}^{A}{}^{B}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\Big|$$
$$\leq\frac{\zeta(m_2,m_1)}{2}\int_0^1\left|\theta^{\alpha}-(1-\theta)^{\alpha}\right|\left|\wp'(m_1+\theta\zeta(m_2,m_1))\right|d\theta$$

By applying Hölder's Integral Inequality,

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^{\alpha}(m_2,m_1)}\right]-\frac{B(\alpha)\Gamma(\alpha)}{2\zeta^{\alpha}(m_2,m_1)}$$
$$\left[{}^{AB}_{m_1}I^{\alpha}_{n}\{\wp(n)\}+{}^{A}{}^{B}_{n}I^{\alpha}_{m_1}\{\wp(m_1)\}\right]\Big|$$
$$\leq\frac{\zeta(m_2,m_1)}{2}\left(\int_0^1\left|\theta^{\alpha}-(1-\theta)^{\alpha}\right|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\left|\wp'(m_1+\theta\zeta(m_2,m_1))\right|^q d\theta\right)^{\frac{1}{q}}\tag{3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.
If we consider $|\wp'|^q$ is supposed to be h-Godunova-Levin preinvex, we obtain

$$\int_0^1|\wp'(m_1+\theta\zeta(m_2,m_1))|^q d\theta\leq\int_0^1\left(\frac{|\wp'(m_1)|^q}{h(\theta)}+\frac{|\wp'(m_2)|^q}{h(1-\theta)}\right)d\theta$$
$$\leq\left(|\wp'(m_1)|^q+|\wp'(m_2)|^q\right)\int_0^1\frac{1}{h(\theta)}d\theta.\tag{4}$$

By using (4) in (3), then it leads to the following result. □

Corollary 3. *There are two possible results are obtained as follows*

1. *If $\alpha=1$ in theorem 5, then we obtain the theorem (3) in [33]*

$$\left|\frac{\wp(m_1)+\wp(m_1+\zeta(m_2,m_1))}{2}-\frac{1}{\zeta(m_2,m_1)}\int_{m_1}^{m_1+\zeta(m_2,m_1)}\wp(\theta)d\theta\right|$$
$$\leq\frac{\zeta(m_2,m_1)}{2(p+1)^{\frac{1}{p}}}(|\wp'(m_1)|^q+|\wp'(m_2)|^q)^{\frac{1}{q}}\left(\int_0^1\frac{1}{h(\theta)}d\theta\right)^{\frac{1}{q}}.$$

2. *Here by taking $\zeta(m_2, m_1) = m_2 - m_1$ and $h(\theta) = \frac{1}{\theta^s}$, we obtain theorem (2.1) introduced by Mudassar in [36]*

$$\left|\frac{\wp(m_1)+\wp(m_2)}{2} - \frac{1}{(m_2-m_1)}\int_{m_1}^{m_2}\wp(\theta)d\theta\right| \\ \leq \frac{(m_2-m_1)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|\wp'(m_1)|^q+|\wp'(m_2)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Corollary 4. *If we consider $\alpha = 1$, $h(\theta) = \theta^s$ ie if $\wp$ is s-Godunova-Levin in theorem 5, then we obtain theorem (3.2) by Noor in [35] as follows*

$$\left|\frac{\wp(m_1)+\wp(m_1+\zeta(m_2,m_1))}{2} - \frac{1}{\zeta(m_2,m_1)}\int_{m_1}^{m_1+\zeta(m_2,m_1)}\wp(\theta)d\theta\right| \\ \leq \frac{\zeta(m_2,m_1)}{2(p+1)^{\frac{1}{p}}}\left[\frac{|\wp'(m_1)|^{\frac{p}{p-1}}+|\wp'(m_2)|^{\frac{p}{p-1}}}{1-s}\right]^{\frac{p-1}{p}}.$$

Theorem 6. *If we consider the assumptions of theorem 5, we get the following inequality related for Hermite-Hadamard inequality as follows*

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^\alpha(m_2,m_1)}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2\zeta^\alpha(m_2,m_1)} \\ \left[{}^{AB}_{m_1}I^\alpha_n\{\wp(n)\} + {}^{A\,B}_{\ n}I^\alpha_{m_1}\{\wp(m_1)\}\right]\Big| \\ \leq \frac{\zeta(m_2,m_1)}{2^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}}\big(|\wp'(m_1)|^q+|\wp'(m_2)|^q\big)^{\frac{1}{q}}\big(1-\frac{1}{2^\alpha}\big)^{1-\frac{1}{q}} \\ \left[\int_0^1\frac{|\theta^\alpha-(1-\theta)^\alpha|}{h(\theta)}d\theta\right]^{\frac{1}{q}}$$

Proof. If we considering lemma 3 and Power-mean inequality, we have

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^\alpha(m_2,m_1)}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2\zeta^\alpha(m_2,m_1)} \\ \left[{}^{AB}_{m_1}I^\alpha_n\{\wp(n)\} + {}^{A\,B}_{\ n}I^\alpha_{m_1}\{\wp(m_1)\}\right]\Big| \\ \leq \frac{\zeta(m_2,m_1)}{2}\int_0^1\big|\theta^\alpha-(1-\theta)^\alpha\big|\big|\wp'(m_1+\theta\zeta(m_2,m_1))\big|d\theta.$$

$$\Big|\frac{\wp(m_1)+\wp(n)}{2}\left[1+\frac{(1-\alpha)\Gamma(\alpha)}{\zeta^\alpha(m_2,m_1)}\right] - \frac{B(\alpha)\Gamma(\alpha)}{2\zeta^\alpha(m_2,m_1)} \\ \left[{}^{AB}_{m_1}I^\alpha_n\{\wp(n)\} + {}^{A\,B}_{\ n}I^\alpha_{m_1}\{\wp(m_1)\}\right]\Big| \\ \leq \frac{\zeta(m_2,m_1)}{2}\left(\int_0^1\big|\theta^\alpha-(1-\theta)^\alpha\big|d\theta\right)^{1-\frac{1}{q}}\left(\int_0^1\big|\theta^\alpha-(1-\theta)^\alpha\big|\big|\wp'(m_1+\theta\zeta(m_2,m_1))\big|^qd\theta\right)^{\frac{1}{q}}.$$

If $|\wp'|^q$ is supposed to be h-Godunova-Levin preinvex, we get

$$\int_0^1 \left|\theta^\alpha - (1-\theta)^\alpha\right| \left|\wp'(m_1 + \theta\zeta(m_2, m_1))\right|^q d\theta \leq \int_0^1 \left|\theta^\alpha - (1-\theta)^\alpha\right| \left(\frac{|\wp'(m_1)|^q}{h(\theta)} + \frac{|\wp'(m_2)|^q}{h(1-\theta)}\right) d\theta$$
$$\leq \int_0^1 \frac{\left|\theta^\alpha - (1-\theta)^\alpha\right|}{h(\theta)} (|\wp'(m_1|^q + |\wp'(m_2)|^q) d\theta$$

Now, by basic calculus,we have

$$\int_0^1 \left|\theta^\alpha - (1-\theta)^\alpha\right| d\theta = \frac{2}{(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)$$

□

Corollary 5. *If $\alpha = 1$, we obtain inequality reported by Ohud-Almutairi and Adem Kiliçman in [33]*

Corollary 6. *If $\zeta(m_2, m_1) = m_2 - m_1$, $h(\theta) = \frac{1}{\theta}$, $q = 1$, and $\alpha = 1$, we have*

$$\left|\frac{\wp(m_1) + \wp(m_2)}{2} - \frac{1}{m_2 - m_1}\int_{m_1}^{m_2} \wp(x)dx\right| \leq \frac{m_2 - m_1}{8}(|\wp'(m_1)| + |\wp'(m_2)|),$$

which is reported by Dragomir and Agarwal in [34].

Corollary 7. *If $\alpha = 1$, $h(\theta) = \theta^s$, we obtain theorem (3.3) by Noor in [35]*

$$\left|\frac{\wp(m_1) + \wp(m_1 + \zeta(m_2, m_1))}{2} - \frac{1}{\zeta(m_2, m_1)}\int_{m_1}^{m_1+\zeta(m_2,m_1)} \wp(\theta)d\theta\right|$$
$$\leq \frac{\zeta(m_2, m_1)}{4}\left[[|\wp'(m_1)|^q + |\wp'(m_2)|^q]\left[\frac{2^{s+1} - 2s}{(s-2)(s-1)}\right]\right]^{\frac{1}{q}},$$

3 Conclusion

Atangana-Baleanu fractional integral operator is found to be very fruitful by researchers in various fields. Its property of reduction to original function when $\alpha = 0$ makes it more powerful and attractive operator as compared to other fractional integral operators.The fruitfulness of AB- operator encouraged us to work with AB- fractional operator in the field of inequalities which opens a new way for the advancement in this field. We have established new version of Hermite-Hadamard type fractional integral inequalities by using Atangana-Baleanu fractional operator (ABFO) for h-Godunova Levin convex and h-Godunova Levin preinvex functions.

Declarations:

Availability of data and material: Not applicable
Competing interests: The authors declare that they have no competing interests.

Authors' contributions:The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.
Funding : None

References

[1] Samraiz, M., Nawaz, F., Iqbal, S., Abdeljawad, T., Rahman, G., & Nisar, K. S. (2020). Certain mean-type fractional integral inequalities via different convexities with applications. Journal of Inequalities and Applications, 2020(1), 1-19.

[2] Peajcariaac, J. E., & Tong, Y. L. (1992). *Convex functions, partial orderings, and statistical applications*. Academic Press.

[3] Dragomir, S. S., & Pearce, C. (2003). Selected topics on Hermite-Hadamard inequalities and applications. *Mathematics Preprint Archive*, 2003(3), 463-817.

[4] Deng, J., & Wang, J. (2013). Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. *Journal of Inequalities and Applications*, **2013**(1), 1-11.

[5] Sarikaya, M. Z., Saglam, A., & Yildirim, H. (2008). On some Hadamard-type inequalities for h-convex functions. J. Math. Inequal, 2(3), 335-341.

[6] Rostamian Delavar, M., Mohammadi Aslani, S., & De La Sen, M. (2018). Hermite-Hadamard-Fejér inequality related to generalized convex functions via fractional integrals. Journal of Mathematics, 2018.

[7] Toader, G. H. (1984). *Some generalizations of the convexity*. In Proc. Colloq. Approx. Optim, Cluj Napoca (Romania),

[8] Qiang, X., Farid, G., Yussouf, M., Khan, K. A., & Rahman, A. U. (2020). New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions. *Journal of Inequalities and Applications*, **2020**(1), 1-13.

[9] Iscan, I., & Wu, S. (2014). Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals. *Applied Mathematics and Computation*, **238**, 237-244.

[10] Ion, D. A. (2007). Some estimates on the Hermite-Hadamard inequality through quasi-convex functions. *Annals of the University of Craiova-Mathematics and Computer Science Series*, **34**, 82-87.

[11] Chen, H., & Katugampola, U. N. (2017). HermiteHadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals. *Journal of Mathematical Analysis and Applications*, **446**(2), 1274-1291.

[12] Huang, C. J., Rahman, G., Nisar K. S., Ghaffar, A., Qi, F. (2019). *Some Inequalities of Hermite-Hadamard type for k-fractional conformable integrals*, Australian Journal of Mathematical Analysis and Applications, 16(1), 1-9.

[13] Nisar, K. S., Rahman, G., Mehrez, K. (2019). Chebyshev type inequalities via generalized fractional conformable integrals, J. Inequal. Appl., 2019:245, https://doi.org/10.1186/s13660-019-2197-1.

[14] Niasr, K.S., Tassadiq, A., Rahman, G., Khan. A. (2019). Some inequalities via fractional conformable integral operators. J. Inequal. Appl., 2019:217, https://doi.org/10.1186/s13660-019-2170-z

[15] Qi, F., Rahman, G., Hussain, S. M., Du, W. S., Nisar, K. S. (2018). *Some inequalities of Čebyšev type for conformable k-fractional integral operators*, Symmetry 2018, 10, 614; doi:10.3390/sym10110614.

[16] Rahman, G., Ullah, Z., Khan, A., Set, E., Nisar, K. S. (2019). *Certain Chebyshev type inequalities involving fractional conformable integral operators*, Mathematics, Mathematics, 7, 364; doi:10.3390/math7040364.

[17] Rahmnan, G., Abdeljawad, T., Jarad, F., Nisar, K. S. (2020). Bounds of Generalized Proportional Fractional Integrals in General Form via Convex Functions and their Applications, Mathematics, 8, 113; doi:10.3390/math8010113

[18] Ali, R. S., Mubeen, S., Nayab, I., Araci, S., Rahman, G., & Nisar, K. S. (2020). Some Fractional Operators with the Generalized Bessel-Maitland Function. *Discrete Dynamics in Nature and Society*, **2020**.

[19] Mubeen, S., & Ali, R. S. (2019). Fractional operators with generalized Mittag-Leffler k-function. *Advances in Difference Equations*, **2019**(1), 520.

[20] Ali, R. S., Mubeen, S., & Ahmad, M. M. (2020). A class of fractional integral operators with multi-index Mittag-Leffler k-function and Bessel k-function of first kind. *J. Math.Computer sci*, **22**(2021), 266-281.

[21] Abdeljawad, T., & Baleanu, D. (2016). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *arXiv preprint arXiv*:1607.00262.

[22] Atangana, A., & Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *arXiv preprint arXiv*:1602.03408.

[23] Rainville, E. D. (1971). Special Functions, *Chelsea Publ. Co., Bronx, New York.*

[24] Dragomir, S. S. (1992). Two mappings in connection to Hadamard's inequalities. Journal of Mathematical Analysis and Applications, 167(1), 49-56.

[25] Almutairi, A., Kiliman, A. (2019). New refinements of the Hadamard inequality on coordinated convex function. Journal of Inequalities and Applications, 2019(1), 1-9.

[26] Dragomir, S. S. (2018). Lebesgue Integral Inequalities of Jensen Type for λ-Convex Functions. Armenian Journal of Mathematics, 10(8), 1-19.

[27] Dragomir, S. S., Agarwal, R. P. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Applied Mathematics Letters, 11(5), 91-95.

[28] Niculescu, C., Persson, L. E. (2006). Convex functions and their applications (pp. xvi+-255). New York: Springer.

[29] Pachpatte, B. G. (2000). On some integral inequalities involving convex functions. RGMIA research report collection, 3(3).Pini, R. (1991). Invexity and generalized convexity. Optimization, 22(4), 513-525.

[30] Tun, M. (2012). On some new inequalities for convex functions. Turkish Journal of Mathematics, 36(2), 245-251.

[31] Peajcariaac, J. E., Tong, Y. L. (1992). Convex functions, partial orderings, and statistical applications. Academic Press.

[32] Almutairi, O., Kiliman, A. (2019). New fractional inequalities of midpoint type via s-convexity and their application. Journal of Inequalities and Applications, 2019(1), 1-19.

[33] Almutairi, O., Kiliman, A. (2019). Some integral inequalities for h-Godunova-Levin preinvexity. Symmetry, 11(12), 1500.

[34] Dragomir, S. S., Agarwal, R. P. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Applied Mathematics Letters, 11(5), 91-95.

[35] Noor, M. A., Noor, K. I., Awan, M. U., Khan, S. (2014). Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions. J. Adv. Math. Stud, 7(2), 12-19.

[36] Muddassar, M., Bhatti, M. I., Iqbal, M. (2012). Some new s-Hermite-Hadamard type inequalities for differentiable functions and their applications. Proc. Pakistan Acad. Sci, 49(1), 9-17.

[37] Sarikaya, M. Z., Set, E., Yaldiz, H., & Basak, N. (2013). HermiteHadamards inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, **57**(9-10), 2403-2407.