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Significant behavior of generalized fractional operators in aspects of k-special functions

Rana Safdar Ali [1,1], *Wajid Iqbal*[1], *Siza Fatima*[2], *Naila Talib*[1], *Humira saif*[1], *Saima Batool*[2]

[1] Department of Mathematics and Statistics, The University of Lahore, Sargodha 40100, Pakistan

[2] University of Agriculture Faisalabad, Faisalabad, Pakistan
Email addresses: safdar.ali@math.uol.edu.pk (R. S. Ali)
wajid.iqbal@math.uol.edu.pk (W. Iqbal)
20nailatalib@gmail.com (N. Talib)
humirasaifuol@gmail.com (H. Saif)
saimabatool1311@gmail.com (S. Batool)

Abstract

The main goal of this paper is to discuss the behavior of generalized fractional operators and its applications with special functions. Moreover, we develop significant results of generalized fractional operators with the power function associated with Bessel function of first kind and visualized in the form of generalized Wright hypergeometric functions. Also, we establish the behavior of generalized these all results discussed in k-fractional calculus.

Keywords: s-type covex function , m-exponential perinvex function , Hölder's inequality, power mean integral inequality

1 Introduction

Special functions have immense applications in the field of applied and pure mathematics. Most of the special functions like as beta function, gamma function, hypergeometric function is not only the series representation but also have integral representation. Special functions have resolved many problems of differential equations and series solutions [2]. Fractional operators can be developed by utilized the different type of special functions as its kernel, which have great applications to develop the fractional calculus [3]. Bessel function is special type series function, which study the solutions of differential equations and they are associated with many problems in the field of mathematical physics, radio physics, nuclear physics and atomic physics. A generalization of the bessel function is also called k-bessel function and also studied in [1].
For the development of our work, we need to remember that following definition we have

[1] Corresponding author

The **left and right sided generalized fractional integral operators** [11] defined for $u_0 > 0$ and $\alpha,\ \beta,\ \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ respectively as

$$(I_{0,u_0}^{\alpha,\beta,\eta}g)(u_0) = \frac{u_0^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^{u_0}(u_0-s)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{s}{u_0})g(s)ds \tag{1}$$

and

$$(I_{u_0,\infty}^{\alpha,\beta,\eta}g)(u_0) = \frac{1}{\Gamma(\alpha)}\int_{u_0}^{\infty}(s-u_0)^{\alpha-1}s^{-\alpha-\beta} \times {}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{u_0}{s})g(s)ds. \tag{2}$$

The **Bessel function** [1] of the first kind defined as

$$J_\nu(t) = \sum_{m=0}^{\infty}\frac{(-1)^m(t/2)^{\nu+2m}}{\Gamma(\nu+m+1)m!} \tag{3}$$

where ν is not a negative integer.
The **Beta function** [13] is defined as

$$\beta(l,m) = \int_0^1 s^{l-1}(1-s)^{m-1}ds, \qquad \Re(l) > 0, \quad \Re(m) > 0 \tag{4}$$

The **Hypergeometric function** [10] defined for $|t| < 1$ by the power series is as

$${}_2F_1(p,q;r;t) = \sum_{n=0}^{\infty}\frac{(p)_n(q)_n}{(r)_n}\frac{t^n}{n!} \tag{5}$$

provided that $r \neq 0,\ -1,\ -2,\ \ldots$. where $(a)_n$ is the Pochhammer symbol.
The **generalized wright hypergeometric function** [8] defined as

$${}_l\psi_h(t) = {}_l\Psi_h\left[\begin{array}{c}(c_i,\alpha_i')_{1,l}\\ (d_j,\beta_j')_{1,h}\end{array}\middle|t\right] \equiv \sum_{m=0}^{\infty}\frac{\prod_{i=1}^{l}\Gamma(c_i+\alpha_i'm)t^m}{\prod_{j=1}^{h}\Gamma(d_j+\beta_j'm)m!} \tag{6}$$

where $t \in \mathbb{C}$, $c_i,\ d_j \in \mathbb{C}$, and $\alpha_i',\ \beta_j' \in \mathbb{R}$ $(i = 1,2,....,l;\ \ j = 1,2,....,h)$.
The **left and right sided generalized k-fractional integral operators** [12] defined for $u_0 > 0$ and $\alpha,\ \beta,\ \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ respectively as

$$(I_{0,u_0}^{\alpha,\beta,\eta}g)_k(u_0) = \frac{u_0^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)}\int_0^{u_0}(u_0-s)^{\frac{\alpha'}{k}-1} \times {}_2F_{1,k}((\alpha+\beta,k),(-\eta,k);(\alpha,k);1-\frac{s}{u_0})g(s)ds \tag{7}$$

and

$$(I_{u_0,\infty}^{\alpha,\beta,\eta}g)_k(u_0) = \frac{1}{k\Gamma_k(\alpha)}\int_{u_0}^{\infty}(s-u_0)^{\frac{\alpha}{k}-1}s^{\frac{-\alpha-\beta}{k}} \times {}_2F_{1,k}((\alpha+\beta,k),(-\eta,k);(\alpha,k);1-\frac{u_0}{s})g(s)ds. \tag{8}$$

The k-**hypergeometric function** [10] defined

$$\forall\ \alpha, \beta, \eta \in \mathbb{C},\ \ \eta \neq 0, -1, -2, -3, \cdots,\ \ |t| < 1$$

by the k-hypergeometric series

$$ {}_2F_{1,k}((\alpha, k), (\beta, k); (\eta, k); t) = \sum_{m=0}^{\infty} \frac{(\alpha)_{m,k}(\beta)_{m,k}}{(\eta)_{m,k}} \frac{t^m}{m!}, \qquad k > 0 \tag{9}$$

where $(\alpha)_{m,k}$, $(\beta)_{m,k}$ and $(\eta)_{m,k}$ are Pochhammer k-symbols.
The **generalized** k-**hypergeometric function** [10] defined as

$$ {}_pF_{q,k}((a_1, k), \cdots, (a_p, k); (b_1, k), \cdots, (b_q, k); t) = \sum_{m=0}^{\infty} \frac{(a_1)_{m,k} \cdots (a_p)_{m,k}}{(b_1)_{m,k} \cdots (b_q)_{m,k}} \frac{t^m}{m!} \tag{10}$$

where a_i, $b_j \in \mathbb{C}$, $b_j \neq 0, -1, \cdots$ $(i = 1, 2, \cdots, p;\ \ j = 1, 2, \cdots, q)$.

The k-**beta function** [4, 6] defined as

$$\beta_k(l, h) = \frac{1}{k} \int_0^1 s^{\frac{l}{k}-1}(1-s)^{\frac{h}{k}-1} ds, \qquad \Re(l) > 0,\ \ \Re(h) > 0 \tag{11}$$

.
The k-**Bessel function** [1] of the first kind defined as

$$J_{\nu,k}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (t/2)^{2m+\frac{\nu}{k}}}{\Gamma_k(\nu + mk + k) m!} \tag{12}$$

where ν is not a negative integer.
The **generalized** k-**Wright hypergeometric function** [7, 8, 9] defined by the series as

$$ {}_l\psi_h^k(t) = {}_l\Psi_h^k \left[\begin{array}{c} (c_i, \alpha_i')_{1,l} \\ (d_j, \beta_j')_{1,h} \end{array} \middle| t \right] \equiv \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{l} \Gamma_k(c_i + \alpha_i' mk) t^m}{\prod_{j=1}^{h} \Gamma_k(d_j + \beta_j' mk) m!} \tag{13}$$

where $k \in \mathbb{R}^+$, $t \in \mathbb{C}$, $c_i, d_j \in \mathbb{C}$, and $\alpha_i', \beta_j' \in \mathbb{R}$ $(i = 1, 2, ..., l;\ \ j = 1, 2, ..., h)$.

Relation between Generalized Fractional Integration and Gamma function

In this section, we discussed a results for left sided generalized fractional integration of a power function in the form of theorems. These results represent a relation between the gamma function and Generalized fractional integration. Also discuss in k-fractional calculus.

Theorem 1. *Let $a, b, c,\ \gamma \in \mathbb{C}$ then hold the following relation*

$$(I_{0,u_0}^{a,b,c}(t_0 - a)^{\gamma-1}(u_0)) = u_0^{-a-b-n}(u_0 - a)^{a+\gamma+n-1} \frac{\Gamma(\gamma)\Gamma(\gamma + c - b)}{\Gamma(\gamma - b)\Gamma(a + \gamma + c)}$$

Proof. Consider the left sided generalized fractional integral operator

$$(I_{0,z}^{a,b,c}h)(z) = \frac{t^{-a-b}}{\Gamma(a)}\int_0^z (z-t)^{a-1}{}_2F_1(a+b,-c;a;1-\frac{t}{z})h(t)dt. \quad (14)$$

Using power function in equation (14), we have

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b}}{\Gamma(a)}\int_0^{u_0} (u_0-t_0)^{a-1}{}_2F_1(a+b,-c;a;1-\frac{t_0}{u_0})(t_0-a)^{\gamma-1}dt_0. \quad (15)$$

Using the equation (5) in equation (15), we obtain

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b}}{\Gamma(a)}\int_0^{u_0}(u_0-t_0)^{a-1} \times \sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}(1-\frac{t_0}{u_0})^n(t_0-a)^{\gamma-1}dt_0$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b}}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}\int_0^{u_0}(u_0-t_0)^{a-1}(1-\frac{t_0}{u_0})^n(t_0-a)^{\gamma-1}dt_0. \quad (16)$$

By putting

$$t_0 = a + z(u_0-a) \implies dt_0 = (u_0-a)dz$$
$$t_0 \to 0 \implies z \to 0$$
$$t_0 \to u_0 \implies z \to 1$$

In the equation (16), we obtain

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b}}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!} \int_0^1((1-z)(u_0-a))^{a-1}(\frac{u_0-a(1-z)}{u_0})^n(z(u_0-a))^{\gamma-1}(u_0-a)dz$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b-n}}{\Gamma(a)}(u_0-a)^{a-1+n+r}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}\int_0^1(1-z)^{a-1}(1-z)^n z^{\gamma-1}dz$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b-n}}{\Gamma(a)}(u_0-a)^{a-1+n+r}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}\int_0^1(1-z)^{a+n-1}z^{\gamma-1}dz.$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}(u_0)) = \frac{u_0^{-a-b-n}}{\Gamma(a)}(u_0-a)^{a-1+n+r}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}\frac{\Gamma(a+n)\Gamma(\gamma)}{\Gamma(a+n+\gamma)}$$

$$\begin{aligned}
(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u_0)) &= \frac{u_0^{-a-b-n}}{\Gamma(a)}(u_0-a)^{a-1+n+r}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a)_n n!}\frac{(a)_n\Gamma(a)\Gamma(\gamma)}{(a+\gamma)_n\Gamma(a+\gamma)}\\
(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u_0)) &= u_0^{-a-b-n}(u_0-a)^{a-1+n+r}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{n!}\frac{\Gamma(\gamma)}{(a+\gamma)_n\Gamma(a+\gamma)}\\
(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u)) &= u_0^{-a-b-n}(u_0-a)^{a-1+n+r}\frac{\Gamma(\gamma)}{\Gamma(a+\gamma)}\sum_{n=0}^{\infty}\frac{(a+b)_n(-c)_n}{(a+\gamma)_n n!}. \qquad (17)
\end{aligned}$$

Using the equation (5) in equation (17), we have

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u_0)) = u_0^{-a-b-n}(u_0-a)^{a-1+n+r}\frac{\Gamma(\gamma)}{\Gamma(a+\gamma)}{}_2F_1(a+b,-c;a+\gamma;1). \qquad (18)$$

since

$${}_2F_1(l,m;n;1) = \frac{\Gamma(n)\Gamma(n-l-m)}{\Gamma(n-l)\Gamma(m-l)}. \qquad (19)$$

Using the equation (19) in equation (18), we have

$$\begin{aligned}
(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u_0)) &= u_0^{-a-b-n}(u_0-a)^{a-1+n+r}\frac{\Gamma(\gamma)}{\Gamma(a+\gamma)}\frac{\Gamma(a+\gamma)\Gamma(a+\gamma-a-b+c)}{\Gamma(a+\gamma-a-b)\Gamma(a+\gamma+c)}\\
(I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}(u_0)) &= u_0^{-a-b-n}(u_0-a)^{a-1+n+r}\frac{\Gamma(\gamma)\Gamma(\gamma+c-b)}{\Gamma(\gamma-b)\Gamma(a+\gamma+c)}.
\end{aligned}$$

□

Theorem 2. *Let a, b, c, $\gamma \in \mathbb{C}$ and k>0, then there relation will be hold*

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma}{k}+n-1}\frac{\Gamma_k(\gamma)\Gamma_k(a+\gamma+c-ak-bk)}{\Gamma_k(a+\gamma-ak-bk)\Gamma_k(a+\gamma+c)}$$

Proof. Consider the left sided generalized k-fractional integral operator with power function

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = \frac{u_0^{\frac{-a-b}{k}}}{k\Gamma_k(\alpha)}\int_0^{u_0}(u_0-t_0)^{\frac{a}{k}-1}{}_2F_{1,k}((a+b,k),(-c,k);(a,k);1-\tfrac{t_0}{z})(t_0-a)^{\frac{\gamma}{k}-1}dt.. \qquad (20)$$

Using the equation (5) in equation (20), we obtain

$$\begin{aligned}
(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) &= \frac{u_0^{\frac{-a-b}{k}}}{k\Gamma_k(\alpha)}\int_0^{u_0}(u_0-t_0)^{\frac{a}{k}-1}\times\sum_{n=0}^{\infty}\\
&\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}(1-\frac{t_0}{u_0})^n(u_0-t_0)^{\frac{a}{k}-1}(t_0-a)^{\frac{\gamma}{k}-1}dt_0. \qquad (21)
\end{aligned}$$

By putting

$$t_0 = a+z(u_0-a) \implies dt_0 = (u_0-a)dz$$

$$t_0 \to 0 \quad \Longrightarrow \quad z \to 0$$
$$t_0 \to u_0 \quad \Longrightarrow \quad z \to 1$$

In the equation (21), we obtain

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = \frac{u_0^{\frac{-a-b}{k}}}{k\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}$$
$$\int_0^1((1-z)(u_0-a))^{\frac{a}{k}-1}\left(\frac{(u_0-a)(1-z)}{u_0}\right)^n(z(u_0-a))^{\frac{\gamma}{k}-1}(u_0-a)dz$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = \frac{u_0^{\frac{-a-b}{k}}}{k\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}\int_0^1(1-z)^{\frac{a}{k}-1+n}(u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}u_0^n z^{\frac{\gamma}{k}-1}dz$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = \frac{u_0^{\frac{-a-b}{k}-n}}{k\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}(u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}\int_0^1(1-z)^{\frac{a+nk}{k}-1}z^{\frac{\gamma}{k}-1}dz. \tag{22}$$

$$= \frac{u_0^{\frac{-a-b}{k}-n}}{\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}(u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}\frac{\Gamma_k(a+nk)\Gamma_k(\gamma)}{\Gamma_k(a+\gamma+nk)}. \tag{23}$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = \frac{u_0^{\frac{-a-b}{k}-n}}{\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}(u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}\frac{(a)_{n,k}\Gamma_k(a)\Gamma_k(\gamma)}{\Gamma_k(a+\gamma+nk)}$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}u_0^{\frac{-a-b}{k}-n}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}\frac{(a)_{n,k}\Gamma_k(\gamma)}{\Gamma_k(a+\gamma+nk)}$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}u_0^{\frac{-a-b}{k}-n}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a)_{n,k}n!}\frac{(a)_{n,k}\Gamma_k(\gamma)}{(a+n)_{n,k}\Gamma_k(a+\gamma)}$$

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}u_0^{\frac{-a-b}{k}-n}\frac{\Gamma_k(\gamma)}{\Gamma_k(a+\gamma)}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(-c)_{n,k}}{(a+\gamma)_{n,k}n!}. \tag{24}$$

Using the equation (9) in equation (24), we have

$$(I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}}u_0^{\frac{-a-b}{k}-n}$$
$$\times\frac{\Gamma_k(\gamma)}{\Gamma_k(a+\gamma)}{}_2F_{1,k}((a+b,k),(-c,k);(a+\gamma,k);1). \tag{25}$$

Since

$$ {}_pF - q21, k(\alpha, 1), (\beta, k)(\eta, k)1 = \frac{\Gamma_k(\eta)\Gamma_k(\eta-\beta-k\alpha)}{\Gamma_k(\eta-k\alpha)\Gamma_k(\eta-\beta)}. \tag{26} $$

Using the equation (26) in equation (25), we get

$$ (I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}} u_0^{\frac{-a-b}{k}-n} \frac{\Gamma_k(\gamma)}{\Gamma_k(a+\gamma)} \frac{\Gamma_k(a+\gamma)\Gamma_k(a+\gamma-ak-bk+c)}{\Gamma_k(a+\gamma-ak-bk)\Gamma_k(a+\gamma+c)} $$

$$ (I^{a,b,c}_{0,u_0}(t_0-a)^{\frac{\gamma}{k}-1})_k(u_0) = (u_0-a)^{\frac{a}{k}-1+n+\frac{\gamma}{k}} u_0^{\frac{-a-b}{k}-n} \frac{\Gamma_k(\gamma)\Gamma_k(a+\gamma-ak-bk+c)}{\Gamma_k(a+\gamma-ak-bk)\Gamma_k(a+\gamma+c)}. $$

□

Remark-I:
If we replace $k=1$ in theorem [2], we have the result show in theorem [1].

Representation in terms of generalized Wright hypergeometric function

In this section, we solve generalized left sided generalized fractional integration for the Bessel functions of first kind, that is represented in terms of generalized Wright hypergeometric function. Also discussed with k in this section.

Theorem 3. *Let a, b, c, γ, $v \in \mathbb{C}$ be such that*

$$ \Re(v) > -1, \;\; \Re(a) > 0, \;\; and \;\; \Re(\gamma+v) > max[0, \Re(b-c)] $$

then there holds the formula

$$ (I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0) = \frac{u_0^{-a-b-n}(u_0-a)^{a+n+\gamma+v-1}}{2^v} \times {}_2\Psi_3\left[\begin{matrix} (\gamma+v,2),(\gamma+c+v-b,2) \\ (\gamma+v-b,2),(\gamma+v+a+c,2),(v+1,1) \end{matrix} \middle| -\frac{(u_0-a)^2}{4}\right]. \tag{27} $$

Proof. Using power function and the equation (7) in left sided generalized fractional integral operator, we have

$$ (I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}J_v(t-a))(u_0) = \frac{u_0^{-a-b}}{\Gamma(a)} \int_0^{u_0}(u_0-t_0)^{a-1}{}_2F_1(a+b,-c;a;1-\frac{t_0}{u_0})(t_0-a)^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{t_0-a}{2})^{v+2n}}{\Gamma(v+n+1)n!}dt_0 $$

$$ (I^{a,b,c}_{0,u_0}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0) = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{v+2m}}{\Gamma(v+n+1)n!} \frac{u_0^{-a-b}}{\Gamma(a)}\int_0^{u_0}(u_0-t_0)^{a-1}{}_2F_1(a+b,-c;a;1-\frac{t_0}{u_0})(t_0-a)^{v+\gamma+2n-1}dt_0 $$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{v+2m}}{\Gamma(v+n+1)n!}\left(I_{0,u_0}^{a,b,c}(t_0-a)^{v+\gamma+2n-1}\right)(u_0). \quad (28)$$

Using the lemma (2.1) as a result and using equation (28) with γ replaced by $\gamma+v+2n$ in equation (28), we get

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{v+2n}}{\Gamma(v+n+1)n!}$$
$$\times u_0^{-a-b-n}(u_0-a)^{a+\gamma+v+2n+n-1}\frac{\Gamma(\gamma+v+2n)\Gamma(\gamma+v+2n+c-b)}{\Gamma(\gamma+v+2n-b)\Gamma(a+\gamma+v+2n+c)}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0)=\frac{u_0^{-a-b-n}(u_0-a)^{a+n+\gamma+v-1}}{2^v}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+v+2n)\Gamma(\gamma+v+2n+c-b)}{\Gamma(\gamma+v+2n-b)\Gamma(a+\gamma+v+2n+c)\Gamma(v+n+1)}\frac{(-1)^n(u_0-a)^{2n}}{2^{2n}n!}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0)=\frac{u_0^{-a-b-n}(u_0-a)^{a+n+\gamma+v-1}}{2^v}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+v+2n)\Gamma(\gamma+v+2n+c-b)}{\Gamma(\gamma+v+2n-b)\Gamma(a+\gamma+v+2n+c)\Gamma(v+n+1)}\frac{(\frac{-(u_0-a)^2}{4})^n}{n!}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\gamma-1}J_v(t_0-a))(u_0)=\frac{u^{-a-b-n}(u_0-a)^{a+n+\gamma+v-1}}{2^v}$$
$$\times\ {}_2\Psi_3\left[\begin{array}{c}(\gamma+v,2),(\gamma+c+v-b,2)\\(\gamma+v-b,2),(\gamma+v+a+c,2),(v+1,1)\end{array}\middle|-\frac{(u_0-a)^2}{4}\right].$$

□

Theorem 4. *Let a, b, c, γ, $v\in\mathbb{C}$ and k>0 be such that*

$$\Re(v)>-1,\ \ \Re(a)>0,\ \ and\ \Re(\gamma+v)>max[0,\Re(b-c)]$$

then there holds the formula

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\frac{u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v}{k}+n-1}}{2^{\frac{v}{k}}}$$
$$\times\ {}_2\Psi_3^k\left[\begin{array}{c}(\gamma+v,2k),(a+\gamma+c+v-ak-bk,2k)\\(a+\gamma+v-ak-bk,2k),(\gamma+v+a+c,2k),(v+k,k)\end{array}\middle|-\frac{(u_0-a)^2}{4}\right]$$

Proof. Using power function and the equation (7) in left sided k-generalized fractional integral operator, we have

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\frac{u_0^{\frac{-a-b}{k}}}{k\Gamma(a)}$$
$$\int_0^{u_0}(u_0-t)^{\frac{a}{k}-1}{}_2F_{1,k}((a+b,k),(-c,k);(a,k);1-\frac{t_0}{u_0})(t_0-a)^{\frac{\gamma}{k}-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{t_0-a}{2})^{\frac{v}{k}+2n}}{\Gamma(v+nk+k)n!}dt_0$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{\frac{v}{k}+2n}}{\Gamma(v+nk+k)n!}\frac{u_0^{\frac{-a-b}{k}}}{k\Gamma(a)}$$
$$\int_0^{u_0}(u_0-t_0)^{\frac{a}{k}-1}{}_2F_{1,k}((a+b,k),(-c,k);(a,k);1-\frac{t_0}{u_0})(t_0-a)^{\frac{\gamma}{k}-1+\frac{v}{k}+2n}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{v+2n}}{\Gamma_k(v+nk+k)n!}\left(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{v+\gamma+2nk}{k}-1}\right)_k(u_0). \tag{29}$$

Using the lemma (2.2) as a result and using equation (29) with γ replaced by $\gamma+v+2nk$ in equation (29), we get

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{v+2n}}{\Gamma_k(v+nk+k)n!}$$
$$u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v+2nk}{k}+n-1}\frac{\Gamma_k(\gamma+v+2nk)\Gamma_k(a+\gamma+v+2nk-ak-bk+c)}{\Gamma_k(a+\gamma+v+2nk-ak-bk)\Gamma_k(\gamma+v+2nk+a+c)}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v+2nk}{k}+n-1}$$
$$\sum_{n=0}^{\infty}\frac{\Gamma_k(\gamma+v+2nk)\Gamma_k(a+\gamma+v+2nk-ak-bk+c)}{\Gamma_k(a+\gamma+v+2nk-ak-bk)\Gamma_k(\gamma+v+2nk+a+c)}\frac{(-1)^n(\frac{1}{2})^{v+2n}}{\Gamma_k(v+nk+k)n!}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\frac{u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v+2nk}{k}+n-1}}{2^{\frac{v}{k}}}$$
$$\sum_{n=0}^{\infty}\frac{\Gamma_k(\gamma+v+2nk)\Gamma_k(a+\gamma+v+2nk-ak-bk+c)}{\Gamma_k(a+\gamma+v+2nk-ak-bk)\Gamma_k(\gamma+v+2nk+a+c)\Gamma_k(v+nk+k)}\frac{(-1)^n(u_0-a)^{2n}}{2^{2n}n!}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\frac{u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v+2nk}{k}+n-1}}{2^{\frac{v}{k}}}$$
$$\sum_{n=0}^{\infty}\frac{\Gamma_k(\gamma+v+2nk)\Gamma_k(a+\gamma+v+2nk-ak-bk+c)}{\Gamma_k(a+\gamma+v+2nk-ak-bk)\Gamma_k(\gamma+v+2nk+a+c)\Gamma_k(v+nk+k)}\frac{-(u_0-a)^2}{4^n n!}$$

$$(I_{0,u_0}^{a,b,c}(t_0-a)^{\frac{\gamma}{k}-1}J_v(t_0-a))_k(u_0)=\frac{u_0^{\frac{-a-b}{k}-n}(u_0-a)^{\frac{a+\gamma+v}{k}+n-1}}{2^{\frac{v}{k}}}$$
$$\times\ {}_2\Psi_3^k\left[\begin{array}{c}(\gamma+v,2\text{k}),(a+\gamma+c+v-a\text{k}-b\text{k},2\text{k})\\(a+\gamma+v-a\text{k}-b\text{k},2\text{k}),(\gamma+v+a+c,2\text{k}),(v\text{+k,k})\end{array}\middle|-\frac{(u_0-a)^2}{4}\right]. \tag{30}$$

□

Remark-II:
When we replace $k=1$ in theorem [4], we get the result show in theorem [3].

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