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Analyzing the Dynamics of Fractional Order Systems: A Topological Approach

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Abstract

This paper predict the topological forms of a mild solution for a control problem governed by semi linear fractional impulsive evolution equations with non local conditions. The $R\delta$ property of the mild solution set gained by applying the steps of non compactness and a fixed point theorem of condensing maps and a fixed point theorem of non convex valued maps.Then this result is applying to the show that the control problem which is done and has a reachable invariant set governed by non linear perturbations. The gained conclusions are also applying to properties the almost controllability of presented control problem.

Keywords: Control problem for semi linear fractional impulsive and non local evolution equation, mild solution, compact $R\delta$ set, measure of non compactness,reachable invariant set, approximate controllability.
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1 Introduction

Fractional calculus gives a best tool to represents the memory by physical and heredity. Fractional order derivated were to be flexible for representing the manners of visco elastic physics, which were successfully applicable to properties of constitutive relations of visco elastic and non-newtonian physics. Fractional derivatives are increasingly used in imaging to take the advantage of nonlocal behavior. physical problems suit in differential equations of fractional order, but they not suited by differential equations. So recently, many researcher have done valuable performance in electro magnetic, control theory, signals, visco elasticity, biological, engineering problems, fluid flows. Hybrid phenomenons is a set of impulsive work with states converges accordance to leminar continuous time delay. In this article we also experience with fractional neutral differential equation including special properties. The existence of mild solution are maintained. The evolution with neutral theory fulfill the much evolutionary changes that exist at molecular level and much of variation between species are because of random genetic variations of mutant that are magnetic electively neutral. The availability of a mild solution to the fractional control issue, which is controlled by semi-linear impulsive and neutral nonlinear systems with non-local circumstances, is the focus of this article."Fractional calculus and its applications in math and alternative Sciences" is dedicated to study the recent works within on top of fields of divisional calculus done by the leading researchers. The mild solution set for the control

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problem governed by the semi linear fractional impulsive and non local evolution equations of the following form:

$$\begin{cases} {}^{C}D_0^{\wp}a(t) = Aa(\ell) + Bu(\ell) + (g(\ell, a(\ell - a), \int_0^{\ell} h(\ell, \kappa, a(\kappa)))d\kappa) \\ \ell \in J = [0, b], \ell \neq \ell_i, i = 1, ..., m, 1 < \wp < 2 \\ \delta a(\ell_i) = I_i(a(\ell_i)), i = 1, ..., m \\ \chi y(0) = a_0 + \omega(a) \quad a'(0) = a_1 \end{cases} \tag{1}$$

The article for this special issue were designated once a careful and studious peer-review method [1].
Mathematical modelling of real-life issues typically ends up in fractional differential equations and numerous alternative issues involving special functions of mathematical physics likewise as their widespread and generalization in single or additional variables. Additionally, most physical processes of fluid, quantum physics, electricity, ecological systems and lots of alternative models are limited at intervals and also their domain of validity is controlled. However, it becomes progressively necessary to be at home with all ancient and recently developed strategies for resolution divisional order PDEs and also applying of thoes methodology.
Fractional calculus may be a branch of Mathematical analysis that studies the many totally different potentialities of process real numbers powers or imaginary number powers of differentiation operators 'D' and of integration operator 'J'. Fractional order differential equations of non-integer order differential equations which may be gained in time and area with power low memory Kernel of non-local relationship.
See official [8], [3], [2], [16], [10] , [5], [6] . The thought of divisional calculus was at first bestowed by Laibniz over three hundred year past.
Recently, the difficulty on the metric form sets for statements and inclusions includes many properties and R_δ condition that are uses by several researchers [9], [7], [4], [12], [13], [14]. We have a tendency to refer the reader to [11], [15], obtained the compactness and R_δ correct of gentle resolution set for an effect drawback mentioned by semi-linear divisional delay evolution equation of the subsequent type. Reachable invariant set is so governable.
However, in some sensible solutions, several natural phenomenans in population dynamics, chemistry, medication and maths could also be study to correct changes like shocks and perturbations. Natural processes can't be delineated by higher order equations. It's natural to correct them by using the inclusions (Semilinear). For your information, there's no proper to examine the R_δ property. Approximate controllability plays a very vital role within the control theory and mathematics due to they are adjacent to the many applications. Specifically, few scintist gave a lot of attention to the approximate controllability for a few types inclusions. [16], [17].

2 Preliminaries

Let$(Z, ||.||)$ is a Banach space. The Banach space $L^p(H, Y)$ denotes the all X-valued Bochner integrable functions and this Banach space has norm of this form $||a||_L^p(H, Y)$. And the Banach space $D(H, Y)$ is the metric space of all Z-valued compact functions and this space has a norm of $||a||_D = \sup_{\ell \in H} ||a(\ell)||$. Another Banach space $QD(H, Y) = Y : H \to Y : a$ is compact. And this space has a norm of $||a||_{MN} = \sup ||a(\ell)||$.
Let X and Y be spaces.we denote as:

$$\begin{aligned} \varrho_{f(c)}(Z) &= \{E \subset X : E \textit{ is a nonempty compact (concave)}\} \\ K_z(Z) &= \{E \subset X : E \textit{ is a nonepmty connect (concave)}\} \end{aligned}$$

For a given multi-valued map, the $G(\varrho)$ is the expressed form of ϱ. If ε is a subset of M then we express the all pre-image of ε by $\varrho^{-}1(\varepsilon)$.

Definition 2.1. [12]. Suppose N be a topological space then
a) If each continuous map $g : E \to M$, E possesses a continuous extension over M, B is a subset of some metric space N, then Q is said to be permenet retract.
b) If M is homeomorphism to the union of a increasing sequence of compact absolute retracts then M is compact absolute retracts.

Lemma 2.1. *ϱ is u.s.c(upper semi continuous) if $\varrho : Z \to P_f(Z)$ is a open and semi-compact multi-valued map.*

Definition 2.2. AR and ANR space

(i) Z is called to be absolute retract space, each continuous function $\nu : E \to Z$ can be explored to a continuous function $\nu : G \to Z$.

(ii) Z is called to be absolute neighborhood retract, $\exists$ a nearby W of E and a explored $\nu : U \to Z$ of ν.

Remark 2.2. *Z is AR Space and also ANR space. Furthermore, suppose E be a AR space. Then E be a concave subset of a commenly concave linear space.*

Definition 2.3. If $\exists$ a point $z_0 \in E$ and a continuous function $g : E \times [0, 1] \to E$ that is $g(z, 1) = z_0$ and $f(z, 0) = z$ for every $z \in E$, then E is contractible.

Definition 2.4. If $\exists$ a increasing sequence L_n of compact non empty contractible sets such that

$$L = \cap_{n=1}^{\infty} L_n$$

then subset L of topological space is said to be $R_\delta - set$.

Remark 2.3. *R_δ set is non empty, compact, connected. The underlaying results are here:*
Compact + convex $\subset$ compact AR-space $\subset$ compact + contractible $\subset$ R_δ-set
and each resulst are good.

Definition 2.5. If ϱ is u.s.c and $\varrho(z)$ is an R_δ-set for every $z \in Z$, then multi-valued map $\varrho : z \to Q_g(W)$.

Remark 2.4. *If each double valued continuous map is R_δ map. surly,*

Theorem 2.5. *Let the $\pi : Y \to Q(Y)$ can be seprated as*

$$\phi = \phi_n o \phi_{n-1} o ... o \phi_1,$$

Suppose Y be a AR space. Here $\emptyset_i : X_{j-1} \to Q(X_j), j, ...m$, are R_δ maps and $X_j, j = 1, ...m - 2$, are ANR-Spaces, and $X_0 = X_n = X$ are AR spaces. ϕ admits a fixed point, if there is a compact subset $E \subset Y$ that is $\phi(Y) \subset E$.

Definition 2.6. $(\geq, \hbar)$ is a partially ordered set and suppose Y be a Banach space. A function $\nu : Q(Y) \to \hbar$ is called to be a MNC in Y if $\nu(bo\varepsilon) = \mu(\varepsilon)$, for each $\varepsilon \in Q(Y)$, Where $bo\varepsilon$ is a ν is called to be:

(i) non singular if $\nu(a \cup \varepsilon) = \nu(\varepsilon)$ for any $a \in X$, $\varepsilon \in Q(Y)$

(ii) monotone if, for each ε_0, $\varepsilon_1 \in Q(Y)$ such that $\varepsilon_0 \subset \varepsilon_1$, one has $\nu(\varepsilon_0) \leq \nu(\varepsilon_1)$

(iii) invariant with reffers to the intersection with a compact set if $\nu(E \cup \varepsilon) = \nu(\varepsilon)$ for each relatively compact subset E of Y and $\varepsilon \in Q(Y)$; If $\hbar$ is a cone in a normed space, we can say that ν is

(iv) regular if $\nu(\varepsilon) = 0$ is equal to the relative compactness of ε.

(v) algebraically half additive if

$$\nu(\varepsilon_0 + \varepsilon_1) \leq \nu(\varepsilon_0) + \nu(\varepsilon_1)$$

for any ε_0, $\varepsilon_1 \in Q(Y)$.

Example of NOC hausdorff NOC : $\gamma(\varepsilon) = \inf\{\epsilon : \varepsilon \ has \ a \ finite \ \epsilon - net\}$ which holds all the above charecteristics.

$$\upsilon(\varepsilon) = max[\gamma(E), mod_C(E)],$$

Where ε is a subset of $PC([0,y];Y)$, $\triangle(\varepsilon)$ represents the collection of all countable subset of ε and

(i) the modaim of equi-continuity

$$mod_c(\varepsilon) = \lim_{\epsilon \to 0} \sup_{a \in \varepsilon} \max_{|\ell_1 - \ell_2| < \epsilon} ||a(\ell_1) - a(\ell_2)||$$

(ii) the damped mod of the fiber compactness

$$\gamma(\varepsilon) = sup_{\ell \in [0,b]} e^{-L\ell} a(\varepsilon(\ell)) \ with \ \varepsilon(\ell) = a(\ell) : a \in \varepsilon$$

the range is of NOC υ is a cone. As proved υ monotone, non singular and regular.

Definition 2.7. A continuous map $F : G \subset Y \to Y$ is prefferd to as condensing with respect to a NOC ν if, for any bounded subset $\varepsilon \subset D$ which is not relatively compact, we gain $\nu(F(\varepsilon))) \neq \nu(\varepsilon)$.

Theorem 2.6. *The fixed point set $Fix(g) = a : a \in g(a)$ is compact. If ε is a bounded concave open subset of a Banach space Y and $G : \varepsilon \to L_z(\varepsilon)$ is a u.s.c ν condensing multi-map.*

Lemma 2.7. *$||u_n||_a \leq \varphi(\ell)$ $\forall$ n=2,3,... and a.e. $\ell \in [0,y]$. The sequence u_n be integrably bounded. Here $\varphi \in L^1([0,a], C^+)$.*

Now we show few fundamentle definitions of the fractional calculus theory.

Definition 2.8. The fractional integral of order $\wp$ with under limit zero for a function $a(\ell)$ is stated as under :

$$I^{\wp} a(\ell) = \frac{1}{\Gamma(\wp)} \int_0^t (\ell - \kappa)^{\wp - 1} a(\kappa) d\kappa, \wp > 0, \ell > 0$$

regarded that the right hand side is point wise stated on $[0, \infty)$, where Γ is the gamma function stated by

$$\Gamma(\wp) = \int_0^\infty \ell^{\wp - 1} e^{-\ell} d\ell$$

Definition 2.9. The R L functional derivative of $\wp$ either the under $a : [0, \infty) \to R$ is stated :

$${}^L D_0^{\wp} a = \frac{1}{\Gamma(n - \wp)} \frac{d^n}{dN\ell} \int_0^t (\ell - \kappa)^{n - \wp - 1} a(\kappa) d\kappa, \quad \ell > 0, \quad 0 \leq m - 1 < \wp < m.$$

Remark 2.8. (i) *The constant with caputo derivative is equal to zero.*

(ii) *The integrals which shows in above definitions are taken in Bochner sense.*

3 Main results

The operators $S_{q(\ell)_{\ell \geq 0}}$ and $\top_{q(\ell)_{\geq 0}}$ are given by

$$S_{\wp}(\ell) = \int_0^\infty \psi_{\wp}(\theta) \top(\theta)(\ell^{\wp}) d\theta$$

$$Q_{\wp}(\ell)a = \wp \int_0^{\infty} \theta\psi_q(\theta)\top(\theta)(\ell^{\wp})d\theta$$

with $\phi_{\wp}$ which is a probability density function and stated on this intervel $(0, \infty)$ as

$$\phi_{\wp}(\vartheta) = \frac{1}{\wp}\vartheta^{-1-\frac{1}{\wp}}\varpi_{\wp}(\theta^{-\frac{1}{q}}) \leq 0,$$

$$\varpi_{\wp}(\vartheta) = \frac{1}{\pi}\Sigma_{n=1}^{\infty}(-1)^{n-1}\vartheta^{-n\wp-1}\frac{\Gamma(n\wp+1)}{n}\sin(n\pi\wp)$$

The above results of R_q and $\top_q$ are used in up coming the mild solution.

4 Mild Solution

$$\begin{cases} {}^C D_0^{\wp} a(t) = Aa(\ell) + Bu(\ell) + (g(\ell, a(\ell - a), \int_0^{\ell} h(\ell, \kappa, a(\kappa)))d\kappa) \\ \ell \in J = [0, b], \ell \neq \ell_i, i = 1, ..., m, 1 < \wp < 2 \\ \delta a(\ell_i) = I_i(a(\ell_i)), i = 1, ..., m \\ \chi y(0) = a_0 + \omega(a) \quad a'(0) = a_1 \end{cases} \tag{2}$$

Lemma 4.1. *If*

$$\begin{aligned} a(\ell) &= a(0) + \chi y'(0)\ell + \frac{A}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}a(\kappa)d\kappa \\ &+ \frac{B}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}u(\kappa)d\kappa + \int_0^t \left(g(\ell, a(\ell-a), \int_0^t h(\ell, \kappa, \chi y(\kappa)))dt\right)d\kappa \end{aligned}$$

holds then we have

$$\begin{aligned} v(\lambda) &= \int_0^t Q_{\wp}(\ell-\kappa)(\ell-\kappa)^{\wp-1}[Bu(\kappa) + (g(\kappa, a(\kappa)), \int_0^t h(\kappa, \ell, a(\ell)))d\kappa \\ &+ S_{\wp}(a_0+\omega) + Q_{\wp}(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1 + \Sigma_{0<\ell_i<\ell}S_{\wp}(\ell-\ell_i)I_i(a(\ell_i))] \end{aligned}$$

Proof.

$$\begin{aligned} {}^C D_0^{\wp} a(\ell) &= Aa(\ell) + Bu(\ell) + g(\ell, a(\ell - \chi y), \int_0^t h(\ell, \kappa, a(\kappa))d\kappa \\ a(\ell) - a(0) - \chi y'(0)\ell &= \frac{A}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}a(\kappa)d\kappa \\ &+ \frac{B}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}u(\kappa)d\kappa \\ &+ \int_0^t \left(g(\ell, a(\ell-a), \int_0^t h(\ell, \kappa, a(\kappa)))dt\right)d\kappa \\ \chi y(\ell) &= a(0) + a'(0)\ell + \frac{A}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}a(\kappa)d\kappa \\ &+ \frac{B}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1}u(\kappa)d\kappa \\ &+ \int_0^t \left(g(\ell, a(\ell-a), \int_0^t h(\ell, \kappa, a(\kappa)))dt\right)d\kappa \end{aligned}$$

taking laplace on both sides

$$\begin{aligned}
\pounds a(t) &= \pounds a(0) + \pounds a'(0)\ell + \frac{A}{\Gamma(\wp)}\pounds\int_0^t (\ell - s)^{\wp-1}\pounds a(\kappa)d\kappa \\
&+\frac{B}{\Gamma(\wp)}\pounds\int_0^t (\ell-\kappa)^{\wp-1}\pounds(u\kappa)d\kappa \\
&+\pounds\int_0^t \left(g(\ell, a(\ell-a), \int_0^t h(\ell,\kappa,a(\kappa)))dt\right)d\kappa
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty e^{-(\lambda)(\kappa)}a(\kappa)d\kappa &= \int_0^\infty e^{-(\lambda)(\kappa)}a(0)d\kappa + \frac{a_1}{\lambda^2} + \frac{A}{\lambda^\wp}\int_0^\infty e^{-(\lambda)(\kappa)}a(\kappa)d\kappa \\
&+\frac{B}{\Gamma(\wp)}\lambda^\wp\int_0^\infty e^{-(\lambda)(\kappa)}u(\kappa)d\kappa \\
&+\int_0^t (g(\ell, a(\ell-a), \int_0^t h(\ell,\kappa,a(\kappa))))d\kappa
\end{aligned}$$

Let

$$\begin{aligned}
v(\lambda) &= \int_0^\infty e^{-(\lambda)(\kappa)}a(\kappa)d\kappa \quad u(\lambda) = \int_0^\infty e^{-(\lambda)(\kappa)}u(\kappa)d\kappa \\
z(\lambda) &= \int_0^\infty e^{-(\lambda)(\kappa)}a(0)d\kappa \quad \omega(\lambda) = \int_0^\infty \left(g((\ell, a(\ell-a)), \int_0^t h(\ell,\kappa,a(\kappa)))dt\right)d\kappa
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= \frac{Av(\lambda)}{\lambda^\wp} + \frac{Bu(\lambda)}{\lambda^\wp} + \frac{z(\lambda)}{\lambda^\wp} + \frac{\omega(\lambda)}{\lambda^\wp} + \frac{a_1}{\lambda^2} \\
v(\lambda) - \frac{Av(\lambda)}{\lambda^\wp} &= \frac{Bu(\lambda)}{\lambda^\wp} + \frac{z(\lambda)}{\lambda^\wp} + \frac{\omega(\lambda)}{\lambda^\wp} + \frac{a_1}{\lambda^2} \\
v(\lambda)[1 - \frac{A}{\lambda^\wp}] &= \frac{[Bu(\lambda) + z(\lambda) + \omega(\lambda)]}{\lambda^\wp} + \frac{a_1}{\lambda^2}
\end{aligned}$$

$$\begin{aligned}
v(\lambda)[\lambda^\wp - A] &= \lambda^\wp\frac{[Bu(\lambda) + z(\lambda) + \omega(\lambda)]}{\lambda^\wp} + \lambda^\wp\frac{a_1}{\lambda^2} \\
v(\lambda) &= Bu(\lambda)[\lambda^\wp - A]^{-1} + z(\lambda)[\lambda^\wp - A]^{-1} + \omega(\lambda)[\lambda^\wp - A]^{-1} \\
&+a_1\lambda^{\wp-2}[\lambda^\wp - A]^{-1}
\end{aligned}$$

put

$$[\lambda^\wp - A^{-1}] = \int_0^\infty e^{(-\lambda)(\kappa)}Q(\kappa)d\kappa$$

$$\begin{aligned}
v(\lambda) &= Bu(\lambda)\int_0^\infty e^{-(\lambda^\wp)(\kappa)}Q(\kappa)d\kappa + z(\lambda)\int_0^\infty e^{-(\lambda^\wp)(\kappa)}Q(\kappa)d\kappa \\
&+\omega(\lambda)\int_0^\infty e^{-(\lambda^\wp)(\kappa)}Q(\kappa)d\kappa + a_1\lambda^{\wp-2}\int_0^\infty e^{-(\lambda^\wp)(\kappa)}Q(\kappa)d\kappa
\end{aligned}$$

Let

$$\begin{aligned}
\kappa &= \ell^{\wp} \\
d\kappa &= \wp\ell^{\wp-1}dt \\
v(\lambda) &= Bu(\lambda)\int_0^\infty e^{-(\lambda^{\wp})(\ell^{\wp})}Q(\ell^{\wp})\wp\ell^{\wp-1}dt + z(\lambda)\int_0^\infty e^{-(\lambda^{\wp})(\ell^{\wp})}Q(\ell^{\wp})\wp\ell^{\wp-1}dt \\
&\quad +\omega(\lambda)\int_0^\infty e^{-(\lambda^{\wp})(t^{\wp})}Q(\ell^{\wp})\wp\ell^{\wp-1}dt + a_1\lambda^{\wp-2}\int_0^\infty e^{-(\lambda^{\wp})(\ell^{\wp})}Q(\ell^{\wp})\wp\ell^{\wp-1}dt
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= Bu(\lambda)\int_0^\infty e^{-(\lambda t)^{\wp}}Q(\ell^{\wp})\wp\ell^{\wp-1}dt + z(\lambda)\int_0^\infty e^{-(\lambda t)^{\wp}}Q(\ell^{\wp})\wp\ell^{\wp-1}dt \\
&\quad +\omega(\lambda)\int_0^\infty e^{-(\lambda\ell)^{\wp}}Q(\ell^{\wp})\wp\ell^{\wp-1}dt + a_1\lambda^{\wp-2}\int_0^\infty e^{-(\lambda\ell)^{\wp}}Q(\ell^{\wp})\wp\ell^{\wp-1}dt
\end{aligned}$$

$$\begin{aligned}
&put \\
e^{-(\lambda\ell)^{\wp}} &= \int_0^\infty e^{-(\lambda)(\ell)(\theta)}\psi_{\wp}(\theta)d\theta \\
v(\lambda) &= Bu(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell)(\theta)}\psi_{\wp}(\theta)Q(\ell^{\wp})\wp\ell^{\wp-1}dtd\theta \\
&\quad +z(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell)(\theta)}\psi_{\wp}(\theta)Q(\ell^{\wp})\wp\ell^{\wp-1}dtd\theta \\
&\quad +\omega(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell)(\theta)}\psi_{\wp}(\theta)Q(\ell^{\wp})\wp\ell^{\wp-1}dtd\theta \\
&\quad +a_1\lambda^{\wp-2}\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell)(\theta)}\psi_{\wp}(\theta)Q(\ell^{\wp})\wp\ell^{\wp-1}dtd\theta
\end{aligned}$$

put

$$\begin{aligned}
\ell &= \frac{\ell'}{\theta} \\
d\ell &= \frac{d\ell'}{\theta} \\
v(\lambda) &= Bu(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\frac{dt'}{\theta}d\theta \\
&\quad +z(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\frac{d\ell'}{\theta}d\theta
\end{aligned}$$

$$\begin{aligned}
&+\omega(\lambda)\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\frac{dt'}{\theta}d\theta \\
&+a_1\lambda^{\wp-2}\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\frac{d\ell'}{\theta}d\theta
\end{aligned}$$

$$v(\lambda) = \int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}Bu(\lambda)\frac{d\ell'}{\theta}d\theta$$

$$+\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_{\wp}(\theta)\frac{Q(\ell'\wp)}{(\theta^{\wp})}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}z(\lambda)\frac{d\ell'}{\theta}d\theta$$

$$+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\omega(\lambda)\frac{dt'}{\theta}d\theta$$
$$+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\chi y_1\frac{dt'}{\theta}d\theta$$

put the values of $u(\lambda)$, $z(\lambda)$, $\omega(\lambda)$

$$\begin{aligned} v(\lambda) &= \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}B\int_0^\infty e^{-(\lambda)(\kappa)}u(\kappa)d\kappa\frac{dt'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}\int_0^\infty e^{-(\lambda)(\kappa)}a(0)d\kappa\frac{dt'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp \\ &\frac{\ell'^{\wp-1}}{\theta^{\wp-1}\int_0^\infty(g(\ell,a(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa))}d\kappa\frac{d\ell'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}a_1\frac{d\ell'}{\theta}d\theta \end{aligned}$$

$$\begin{aligned} v(\lambda) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell'+\kappa)}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}Bu(\kappa)d\kappa\frac{d\ell'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell'+\kappa)}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}a(0)d\kappa\frac{d\ell'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}} \\ &\int_0^\infty(g(\ell,a(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa)))d\kappa\frac{d\ell'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{t'^{\wp-1}}{\theta^{\wp-1}}a_1\frac{d\ell'}{\theta}d\theta \end{aligned}$$

$$\begin{aligned} v(\lambda) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}Bu(\kappa)d\kappa\frac{dt'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}a(0)d\kappa\frac{d\ell'}{\theta}d\theta \end{aligned}$$

$$\begin{aligned} &+\int_0^\infty \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}} \\ &\int_0^\infty(g(\ell,\chi y(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa)))d\kappa\frac{d\ell'}{\theta}d\theta \\ &+\int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')}\psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^{\wp-1}}a_1\frac{d\ell'}{\theta}d\theta \end{aligned}$$

$$\begin{aligned} v(\lambda) &= \int_0^\infty e^{-(\lambda)(\ell)}\Big[\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{\theta^\wp}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}Bu(\kappa)d\kappa d\ell' \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{\theta^\wp}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}a(0)d\kappa d\ell' \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp \end{aligned}$$

$$\begin{aligned} &\frac{\ell'^{\wp-1}}{\theta^\wp}\int_0^\infty (g(\ell,a(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa))d\kappa)d\ell' \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}a_1 d\ell'\Big]d\theta \end{aligned}$$

Now we invert laplace transform

$$\begin{aligned} v(\lambda) &= \Big[\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}Bu(\kappa)d\kappa d\ell' \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}a(0)d\kappa d\ell' \\ &+\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^\wp} \\ &\int_0^\infty (g(\ell,a(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa)))d\kappa d\ell' \\ &+\int_0^\infty \psi_\wp(\theta)\frac{Q(\ell'\wp)}{(\theta^\wp)}\wp\frac{\ell'^{\wp-1}}{\theta^\wp}a_1 d\ell'\Big] \\ v(\lambda) &= \Big[\int_0^\infty\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell^\wp)\wp(\ell^{\wp-1})\theta dt Bu(\kappa)d\kappa \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell^\wp)\wp(\ell^{\wp-1})\theta d\ell a(0)d\kappa \\ &+\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell^\wp)\wp(\ell^{\wp-1})\theta d\ell \\ &\int_0^\infty (g(\ell,a(\ell-a)),\int_0^t h(\ell,\kappa,a(\kappa)))d\kappa \\ &+\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell^\wp)\wp(\ell^{\wp-1})\theta a_1 d\ell\Big] \end{aligned}$$

$$Q_\wp(\ell)=\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell^\wp)\wp(\ell^{\wp-1})\theta dt$$

$$\begin{aligned} v(\lambda) &= \Big[\int_0^\infty\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell-\kappa)^\wp\wp(\ell-\kappa)^{\wp-1}\theta d\ell Bu(\kappa)d\kappa \\ &+\int_0^\infty\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell-\kappa)^\wp\wp d\ell a(0)(\ell-\kappa)^{\wp-1}\theta d\kappa \\ &+\int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell-\kappa)^\wp\wp\ell^{\wp-1}\theta dt \end{aligned}$$

$$\int_0^\infty (g(\ell, a(\ell - a)), \int_0^t h(\ell, \kappa, a(\kappa)))d\kappa + \int_0^\infty \psi_\wp(\theta)Q(\theta)(\ell-\kappa)^\wp \wp(\ell^{\wp-1})\theta a_1 d\ell\Big]$$

$$\begin{aligned} v(\lambda) &= \Big[\int_0^t Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}Bu(\kappa)d\kappa + \int_0^t Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a(0)d\kappa \\ &+ \int_0^t Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}d\ell \int_0^s (g(\ell, a(\ell-a)), \int_0^t h(\ell,\kappa,a(\kappa)))d\kappa \\ &+ \int_0^t Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1 d\ell\Big] \end{aligned}$$

$$\begin{aligned} v(\lambda) &= \int_0^t Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}[Bu(\kappa) + (g(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell)))d\kappa \\ &+ S_\wp(a_0+\omega) + Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1 + \Sigma_{0<\ell_i<\ell}S_\wp(\ell-\ell_i)I_i(a(\ell_i))] \end{aligned}$$

□

This is a mild solution of control problem (1.2)

5 Example

Example 5.1. Let $P\wp > 1$. Let $Q_\wp(\ell)$ be uniformly continuous for $\ell > 0$. We take $v, \nu(v)$ is non empty ; if, in addition, $R_\wp(\ell)$ is compact for $\ell > 0$ then $\nu(v)$ *is an* R_δ set.

Proof. The operator F^u is stated as on $OD(H, A)$ as

$$\begin{aligned} F^u(a)(\ell) &= S_\wp(\ell)(a_0 + \omega(a)) + \int_0^t (\ell - s)^{\wp-1}Q_\wp(\ell-\kappa) \\ &\Big[Bu(\kappa) + (g(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell)\Big]d\kappa \\ &+ Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1 + \Sigma_{0<\ell_i<\ell}S_\wp(\ell-\ell_i)I_i(a(\ell_i))) \end{aligned}$$

We make four sections of this proof. In common sense, firstly we proof first part of this theorem. That is exactlly that $a \in \nu(v)\ \forall\ E^v$ has a fixed point . We states that $E^v = F_1 + F_2^u$ here

$$\begin{aligned} F_1(a)(\ell) &= S_\wp(\ell)(a_0+\omega(a)) + Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1 + \Sigma_{0<\ell_i<\ell}S_\wp(\ell-\ell_i)I_i(a(\ell_i)) \\ F_2^u(a)(\ell) &= \int_0^t (\ell-\kappa)^{\wp-1}Q_\wp(\ell-\kappa)\Big[Bu(\kappa) + (g(\kappa,a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell)\Big]d\kappa \end{aligned}$$

STEP 1. A priori bounded solution. such that $||a|| \leq r$

$$\begin{aligned}
||a(\ell)|| &\leq ||S_{\wp}(\ell)a_0|| + ||S_{\wp}(\ell)\omega(a)|| + \int_0^t (\ell-\kappa)^{\wp-1}||Q_{\wp}(\ell-\kappa)Bu(\kappa)|| \\
&+ \int_0^t (\ell-\kappa)^{\wp-1}||Q_{\wp}(\ell-\kappa)(g(\kappa, a(\kappa)), \\
&\int_0^t h(\kappa, \ell, a(\ell))d\ell)||d\kappa \\
&+||Q_{\wp}(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1|| + ||\Sigma_{0<\ell_i<\ell}S_{\wp}(\ell-\ell_i)I_i(a(\ell_i))||
\end{aligned}$$

$$\begin{aligned}
||a(\ell)|| &\leq N||a_0|| + N||\omega(A)|| + \frac{\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}||Bu(\kappa)||d\kappa \\
&+N\Sigma_{0<\ell_i<\ell}d_i||(a(\ell_i))|| + Nm||I_i(0)|| + \frac{\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1} \\
&\left[\mu_1(\kappa)||a(\kappa)|| + \mu_1(\kappa)\int_0^{\kappa} q(\kappa,\ell)||a(\ell)||d\ell\right]d\kappa \\
&+\frac{\wp N}{\Gamma(\wp+1)}(\ell-\kappa)^{\wp-1}||a_1||
\end{aligned}$$

$$\begin{aligned}
||a(\ell)|| &\leq N||a_0|| + NM^* + \frac{\wp N b^{\wp} - \frac{1}{p}}{\Gamma(\wp+1)}\left(\frac{P-1}{p\wp-1}\right)^{\frac{p-1}{p}}||Bu|| \\
&+N\Sigma_{0<\ell_i<\ell}d_i||(a(\ell_i))|| \\
&+Nm||I_i(0)|| + \frac{\wp N}{\Gamma(\wp+1)}\sup\mu_1(\kappa)(1+||q_*||_L^{\infty})\int_0^t (\ell-\kappa)^{\wp-1}||a(\kappa)||d\kappa \\
&+\frac{\wp N}{\Gamma(\wp+1)}(\ell-\kappa)^{\wp-1}||a_1||
\end{aligned}$$

Let

$$\begin{aligned}
c_1 &= \frac{\wp N}{\Gamma(\wp+1)}\sup\mu_1(\kappa)(1+||q_*||_L^{\infty})\int_0^t (\ell-\kappa)^{\wp-1}||a(\kappa)||d\kappa \\
&+\frac{\wp M}{\Gamma(\wp+1)}(\ell-\kappa)^{\wp-1}||a_1|| \\
c_2 &= N||a_0|| + NM^* + \frac{\wp N b^{\wp} - \frac{1}{p}}{\Gamma(\wp+1)}(\frac{P-1}{p\wp-1})^{\frac{p-1}{p}}||Bu|| \\
&+Nm||I_i(0)|| + N\Sigma_{i=1}^m d_i||(a(\ell_i))|| \\
||a(\ell)|| &\leq c_1[1+H^*E_{\wp}(c_2\Gamma(\wp)b^{\wp})]^i E_{\wp}(c_2\Gamma(\wp)b^{\wp})
\end{aligned}$$

This implies that $||a|| \leq s$.
in $PC(J,X)$. We obtain $h(\kappa,\ell,a_n(\ell)) \to h(\kappa,\ell,a(\ell))$, a.e.$(\kappa,\ell) \in \Xi$

$$||h(\kappa,\ell,a_n(\ell))|| \leq q(\kappa,\ell)||a_n(\ell)|| \leq q(\kappa,\ell)ra.e.(\kappa,\ell) \in \Xi$$

Hence, from the lebesgue dominated convergence theorem, we can deduce

$$\int_0^{\kappa} h(\kappa,\ell,a_n(\ell))d\ell \to \int_0^{\kappa} h(\kappa,\ell,a(\ell))d\ell, a.e.(\kappa,\ell) \in \Xi$$

$$\begin{aligned}
&\int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa)(g(\kappa, a_n(\kappa)), \int_0^\kappa h(\kappa,\ell,a_n(\ell))d\ell)d\kappa \\
&\to \int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa)(g(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell)d\kappa
\end{aligned}$$

$$\begin{aligned}
\leq\ & M||\omega(a_n)-\omega(a)|| + \Sigma_{i=1}^m M||I_i(a_n(\ell)) - I_i(a(\ell))|| \\
&+|| \int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa)[(g(\kappa, a_n(\kappa)), \int_0^\kappa h(\kappa,\ell,a_n(\ell))d\ell) \\
&-(g(\kappa, a(\kappa)), \int_0^\kappa h(\kappa,\ell,a(\ell))d\ell)]d\kappa \to 0
\end{aligned}$$

This shows the E^v is compact.

STEP 3. E^v *is* μ - condensing. Let $\mho \in OD(H, A)$ be a convergence set that is

$$\nu(F^u(\Omega)) \geq \nu(\Omega).$$

However, we take a series $a_n \subset \mho$ hold

$$\begin{aligned}
z_n(\ell) &= S_\wp(\ell)(a_0+\omega(a)) + \int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa) \\
&\quad \left[Bu(\kappa) + (g(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell)\right] d\kappa \\
&\quad + Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1} a_1 + \sigma_{i=1}^m S_\wp(\ell-\ell_i) I_i(a(\ell_i)) \\
\nu(F^u(\Omega)) &= [\gamma(z_n), mod_C(z_n)]
\end{aligned}$$

Connectedness of series ω, I_i, we gain

$$X(S_\wp(\ell)\omega(a_n)) = 0, \quad X(S_\wp(\ell-\ell_i)I_i(a(\ell_i))) = 0$$

$$(\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa)(g(\kappa, a_n(\kappa)), \int_0^\kappa h(\kappa,\ell,a_n(\ell))d\ell$$

$$\begin{aligned}
X(z_n(\ell)) &= X(S_\wp(\ell)(a_0+\omega(a)) + \int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa) \\
&\quad \left[Bu(\kappa) + (g(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell\right] d\kappa \\
&\quad + Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1} a_1 + \Sigma S_\wp(\ell-\ell_i) I_i(a(\ell_i)))
\end{aligned}$$

$$\begin{aligned}
\leq\ & X(S_\wp(\ell)\omega(a_n)(\ell)) + \Sigma X(S_\wp(\ell-\ell_i)I_i(a(\ell_i))) \\
&+X(Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1) \\
&+X(\int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa) g(\kappa, a_n(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell d\kappa) \\
&+X(Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1}a_1)
\end{aligned}$$

$$\leq \frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\kappa-\ell)^{\wp-1}\mu_2(\kappa)\Big[X(a_n(\kappa))+X\Big(\int_0^t h(\ell,\kappa,a(\ell))d\ell\Big)\Big]d\kappa$$
$$\leq \frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}\mu_2(\kappa)\Big[X(a_n(\kappa))+2\Big(\int_0^\kappa \mu_3(\kappa,\ell)d\ell\Big)X(a_n(\kappa))\Big]d\kappa$$

$$\leq \frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}\mu_2(\kappa)(1+2\mu_3^*)e^{L\kappa}e^{-L\kappa}X(a_n(\kappa))]d\kappa$$
$$\leq \frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}\mu_2(\kappa)(1+2\mu_3^*)e^{L\kappa}\mu_2(\kappa)\sup e^{-L\kappa}X(a_n(\kappa))]d\kappa$$
$$= \frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}\mu_2(\kappa)(1+2\mu_3^*)e^{L\kappa}\mu_2(\kappa)d\kappa\gamma(a_n)$$

Now we write
$$\sigma(\ell)=\frac{2\wp N}{\Gamma(\wp+1)}\int_0^t (\ell-\kappa)^{\wp-1}\mu_2(\kappa)(1+2\mu_3^*)e^{L\kappa}\mu_2(\kappa)d\kappa$$
and choice$L>0$ enoughly big that is
$\max e^{-L\ell}\sigma(\ell)<1$
we assuming that

$$\gamma(a_n)\leq\gamma(z_n)=supe^{-L\ell}X(z_n(\ell))\leq supe^{-L\ell}\sigma(\ell)\gamma(a_n)$$

This goes further $\gamma(a_n)=0$
On the contrary , we read
$$g_n(\ell)=f(\ell,a_n(\ell),\int_0^t h(\ell,\ell,a_n(\ell))d\ell)+Bu(t)$$

Furthermore, in the form of strong compactness of $\top(\ell)$ and the conectedness of $\mho$, I_i, we proof easily this that the set F_1a_n is relatively correct. thus, since three sets F_1a_n and $F_2^ua_n$ are relatively compact, we have

$$mod_C(z_n)=mod_C(F^ua_n)=mod_C(F_1a_n+F_2^ua_n)=0$$

$$F_2^ua_n(\ell)=\int_0^t(\ell-\kappa)^{\wp-1}Q_\wp(\ell-\kappa)g_n(\kappa)d\kappa$$

is equi-continuous.
Therefore the set F_1a_n is relatively compact. combining the above equations , some has $\upsilon(\Omega)=0$ we refers that E^v is connectedness. We proofs that the fixed point set for F^u is non empty and compact. suppose $\rho\in OD(H;A)$ be the answer of differential equation

$$\begin{aligned}||\rho(\ell)|| &= N||a_0||+NM^*+\frac{\wp Nb^{\wp-\frac{1}{p}}}{\Gamma(\wp+1)}\frac{P-1}{p\wp-1}^{\frac{p-1}{p}}||Bu||\\ &+M\Sigma_{i=m}^0 d_i||(\xi(\ell_i))||+Nm||I_i(0)||\\ &+\frac{\wp N}{\Gamma(\wp+1)}\sup\mu_1(\kappa)(1+||q_*||_L^\infty)\int_0^t(\ell-\kappa)^{\wp-1}||\xi(\kappa)||d\kappa\\ &+\frac{\wp N}{\Gamma(\wp+1)}(\ell-\kappa)^{\wp-1}||\xi_1||\end{aligned}$$

Then it is easy to verify that $\mho$ is compact, open and concave set. The closed ness of $\mho$ together with relatively connectedness of $\mho$ and compactness of F^u refers that F^u has compact values . However, if $a\in\mho$, then alike to the results .we have

$$\begin{aligned}
E^v(a)(\ell) &\leq N||a_0|| + NM^* + \frac{\wp M b^{\wp-\frac{1}{p}}}{\Gamma(\wp+1)} \frac{P-1}{p\wp-1}^{\frac{p-1}{p}} ||Bu|| \\
&+ N\Sigma_{i=m}^{0} d_i||(a(\ell_i))|| + Nm||I_i(0)|| \\
&+ \frac{\wp N}{\Gamma(\wp+1)} \sup \mu_1(\kappa)(1+||q_*||_L^\infty) \int_0^t (\ell-\kappa)^{\wp-1} \sup ||a(\ell)|| d\kappa \\
&+ \frac{\wp N}{\Gamma(\wp+1)} (\ell-\kappa)^{\wp-1} ||a_1||
\end{aligned}$$

$$\begin{aligned}
F^u(a)(\ell) &\leq M||a_0|| + NM^* + \frac{\wp M b^{\wp-\frac{1}{p}}}{\Gamma(\wp+1)} \frac{P-1}{p\wp-1}^{\frac{p-1}{p}} ||Bu|| \\
&+ N\Sigma d_i||(a(\ell_i))|| + Nm||I_i(0)|| \\
&+ \frac{\wp M}{\Gamma(\wp+1)} \sup \mu_1(\kappa)(1+||q_*||_L^\infty) \int_0^t (\ell-\kappa)^{\wp-1} \sup ||\xi(\kappa)|| d\kappa \\
&+ \frac{\wp M}{\Gamma(\wp+1)} (\ell-\kappa)^{\wp-1} ||\xi(\kappa)|| \\
F^u(a)(\ell) &= \Xi(\ell)
\end{aligned}$$

STEP 4. As in our knowledge that non linear function f is compact, thus we take a series g_n of locally lips chitz function such that

$$||g_n(\ell, a(\ell), \int_0^t h(\ell,\ell,a(\ell))d\ell) - g(\ell, a(\ell), \int_0^t h(\ell,\ell,a(\ell))d\ell)|| < \epsilon_n$$

suppose we state an infinitesimal operator by

$$\begin{aligned}
F_n^u(a)(\ell) &= S_\wp(\ell)(a_0 + \omega(a)) + \int_0^t (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa) \\
&\left[Bu(\kappa) + (g_n(\kappa, a(\kappa)), \int_0^t h(\kappa,\ell,a(\ell))d\ell)\right] d\kappa \\
&+ Q_\wp(\ell-\kappa)(\ell-\kappa)^{\wp-1} a_1 + \Sigma_{0<\ell_i<\ell} S_\wp(\ell-\ell_i) I_i(a(\ell_i)))
\end{aligned}$$

for every $a \in OD(H, A)$. Indeeed, the operator F_n^u is well defined. We too note that for every $a \in OD(H, A)$

$$\begin{aligned}
&\leq \frac{2\wp N}{\Gamma(\wp+1)} \int_0^t (\ell-\kappa)^{\wp-1} ||g_n(\ell, a(\ell), \int_0^t h(\ell,\ell,a(\ell))d\ell) \\
&-g(\ell, a(\ell), \int_0^t h(\ell,\ell,a(\ell))d\ell)|| d\kappa \\
&\leq \frac{\wp N b^\wp}{\Gamma(\wp+1)} \epsilon_n
\end{aligned}$$

$||(I - F_n^u)(a)(\ell) - (I - F^u)(a)(\ell)||$ $I - F_n^u$ reaches to $K - E^v$ uniformly on $OD(H, A)$, Here K represents an identical operator. Moreover

$$\begin{gathered}
||g_n(\ell, a(\ell), \textstyle\int_0^t h(\ell,\kappa,a(\kappa))d\kappa)|| \leq 1 + \mu_1(\ell)(1+||q_*||)||a|| \\
g_n(\kappa) = (\ell-\kappa)^{\wp-1} Q_\wp(\ell-\kappa) g(\kappa, a_k(\kappa), \textstyle\int_0^s h(\kappa,\ell,a_k(\ell))d\ell)
\end{gathered}$$

is derivatively bounded for every bounded series $a_k \subset OD(H, A)$.Thus it follows from the connectedness of

$R_{\wp}(\ell)$ for $\ell > 0$ that $X(g_n(s)) = 0$ for every $\ell \in K$, $\kappa < \ell$

$$(I - F_n^u)(a) = y$$

has at least a mild solution for every $y \in OD(H, A)$. Now the locally lipschitz continuity of g_n implies that the solution is unique. □

6 Conclusion

In above thesis, mild solution of the system of fractional evolution equations is find out with renewed and to practice of existing releted literature. The underlaying have been with logics and characteristics:

1. The topological structure of mild solution set for a control problem controled by semi linear fractional impulsive evolution equations by non local conditions.
2. The $R\delta$ Property of mild solution set is gained by put the measure of non compactness and a fixed point theorem of non convex valued graph.
3. Then this conclusions is put to prove that the presented control problem has a reachable invariant set under non linear perturbation.
4. The gain conclusions are also put to properties the approximate controllability of present control problem.

7 Author's Contributions

All author's contributed equally to the writing of this paper. All author's read and approved the final manuscript.

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