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Topological Techniques for Studying Solution Sets of Fractional Neutral Delay Differential Equations

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Abstract

This article investigates the topological structure set of all mild solutions. The equation has been discussed in this article is Fractional constant evolution equation with finite delay on half line. Our purpose is to show that our solution set is an R_δ set. It was proved on compact intervals. The compact intervals was made by satisfying a result on topological forms of fixed point set by using the krasnosel skii type operators. And at the end, we apply the inverse limit method. By using this method we get the conclusions on half line.

Keywords: Half line, real line, fractional differetial equation, compact intervals, krasnosel skii type operators, inverse limit method, R_δ set. **2019 MSC:** 26A33, 34K37.

1 Introduction

Norm of a Banach space F is $|.|$. Analytic semigroup of operators on F is $\top(\ell)_{\ell\geq 0}$ and the infinitesimal generator of this semi group is $B : D(B) \subset E \to E$. The $\tau : [-\kappa, 0] \to F$ is a function belonging to phase space $C_0 = C([-\kappa, 0]; F)$.

In this article, we proof the existence of results on the half line and specially, we find mild solution of the task is an R_δ set. The aim of this article is to study the topological structure of the set of all mild solutions

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of fractional neutral evolution equations with finite delay of the form:

$$\begin{cases} {}^{C}D^{\alpha}\Big[a(\ell) - h(\ell, a(\ell), D^{\beta}a(\ell))\Big] = Aa(\ell) + f(\ell, a(\ell), D^{\beta}a(\ell)), \ell > 0 \\ a(0) = a_0 \quad a'(0) = a_1, \\ 1 < \alpha < 2, \quad 0 < \beta < 1 \end{cases} \tag{1}$$

R_δ set is a analytic with respect to Cech homology function, i.e. cech homology has the similer homology as the one point space and R_δ set is nonempty compact connected space. It is equivalent to point [14] by order to understand of algebraic topology, it is also a singleton.

It is well famous that, in 1890, cauchy problems are proved by peano.

$$\begin{cases} x'(\ell) = f(\ell, x(\ell)), 0 < \ell \leq a \\ x(0) = x_0 \end{cases}$$

Here f has local solution and has the uniqueness property which did not hold in general, and $f : [0, a] \times R^n \to R^n$ is continuous. These are encourage for studying the structure of solutions. Peano himself proved these, when n=1 then solution becomes

$$\zeta(\ell) = x(\ell) : x \in \zeta$$

is compact, connected, for ℓ in few nearby of 0. In 1925, these conclusions are simplified by kneser into the part of constant m. Further, in 1926, ζ is a continuum having norm proved by Hukuhara.

A much specific feature of this ζ has been proved in 1949 by [2]. They showed that ζ is an R_δ set, so ζ is analytic. The similar results are gained by [8]. For further detail, previous results and relavent references [1], [2], [3].

In more cases, differential problems with solution set often correspond by fixed point set of operators in electible function spaces. The conclusions of fixed point sets on R_δ property mostly gains operators with compactness in all linear space. This application is difficult. In this chapter, we first gain the theorem that fixed point set with R_δ property of krasnosel skii kind of this form $V + B$, where B is contraction and V is compact.

Then by using the inverse limit method and this result, we gain the R_δ property for mild solution set of above equation but also gain the existence result.

We refer the readers [5], [9], [10], [11], [12], [22], [23] and on topological structure of solution sets we focus to [3], [6], [7], [15], [14], [17], [18], [23].

2 Preliminaries

We remember $a : [0, +\infty) \subset S \to F$ then , the order with $\wp > 0$ with under limit zero for a is stated as

$$D^{\wp}a(\ell) = \frac{1}{\Gamma(\wp)} \int_0^t (\ell - \kappa)^{\wp - 1} a(\kappa) d\kappa, \ell > 0$$

and the order of fractional caputo derivative $\wp > 0$ for the function a is stated as

$$^{C}D^{\wp}a(\ell) = I^{n-\wp}a^{(m)}(\ell) = \frac{1}{\Gamma(m-\wp)}\int_0^t (\ell-\kappa)^{n-\wp-1}a^{(m)}(\kappa)d\kappa,$$

regarded as the right hand side of the upper equation are stated as $[1,0)$. We can write the integrals that are represented in above equations are given in bochner point of view. By the generator B, we suppose that it is the infinitesimal operator of an analytic semi group $S(\ell)_{\ell\geq 0}$ such that $0 \in \varrho(D)$. Here $\varrho(D)$ is B. It is called for every that for each $\tau \in (1,0]$, the fractional power A^{τ} is defined by domain of open leminar on $E(A^{\eta})$. The underlaying characteristics will be used

(i) It is a permenent $N \geq 1$ that is

$$N = \sup_{t\in R_+} |S(\ell)| < \infty$$

(ii) The norm of banach space $E(A^{\eta})$ is $||a||_{\beta} = |A^{\beta}a|$, for $a \in C(A^{\beta})$.

(iii) $S(\ell) : F \to C \; \forall \; (A^{\beta}), \forall \; \ell \geq 0$.

(iv) $S(\ell)Ba = B^{\beta}aS(\ell)$, for each $a \in E(A^{\beta})$ and $\ell \geq 0$

(v) There exist $C_{\beta} > 0$ such that $|B^{\beta}S(\ell)| \leq \frac{C_{\beta}}{\ell^{n}}$, $A^{\beta}T(\ell)$ is bounded on E, for every $\ell > 0$.

Firstly we state following lemmas, and we will omit them

Lemma 2.1. *$a, b \in X_{m,n}$, for every $n \in N_m$, we have*

(i)

$$|a^{\varphi}(\ell) - b^{\varphi}(\ell)| \leq 2p_{m,n}(a-b), \forall \;\; \ell \in [-r,n],$$

(ii)

$$||a^{\varphi}_{\ell}|| \leq 2p_{m,n}(a) + ||\varphi||_m, for \;\; all \;\; \ell \in [0,m],$$

(iii)

$$||a^{\varphi}_{\ell}(\ell) - b^{\varphi}_{\ell}||_0 \leq 2p_{m,n}(a-b), \forall \;\; \ell \in [0,n]$$

(iv)

$$||a_{\varphi}|| \leq 2p_{m,n}(a) + ||\varphi||_m$$

Definition 2.1. Let $\varrho : z \times \varrho$ is denoted as

(i) upper semi-continuous $a_0 \in Y$, if for any neighbourhood $O(\varrho(a_0))$ of $\varrho(a_0)$, then $\exists$ a neighbourhood $O(a_0)$ of a_0 that is $\varrho(a) \subset O(\varrho(a_0)) \; \forall \; a \in O(a_0)$

(ii) linear- compact if $\varrho(E)$ is E of Z. If $W \cap F \neq \emptyset$, then $z \in F \cap Z$ is called ϱ if $z \in \varrho(z)$. The set of all fixed point of ϱ is expressed by $Fix(\varrho)$.

Lemma 2.2. *The function R_q and $\top_q$ underlaying condition:*

(i) *For any fixed $\ell \geq 0$, $R_q(\ell)$ and $\top_q(\ell)$ are laminer and compact function. Furthermore, $\forall\ a \in F$, we have*

$$|R_q(\ell)a| \leq M|a| \quad and \quad |\top_q(\ell)a| \leq \frac{qM}{\Gamma(1+q)}|a|$$

(ii) *Operators $S_q(\ell)_{\ell\geq 0}$ and $\top_q(\ell)_{\ell\geq 0}$ are strongly continuous, which means that $\forall\ a \in E$ and $0 \leq \ell_1 < \ell_2$ we have $|R_q(\ell_1)a - R_q(\ell_2)a| \to 0$ and $|\top_q(\ell_1)a - \top_q(\ell_2)a| \to 0$ as $\ell_1 \to \ell_2$*

(iv) *For every $a \in F$; $\wp \in (0,1)$, $\nu \in (0,1]$, we have*

$$A\top_q(\ell)a = A^{1-\nu}\top_q(\ell)A^{\nu}a, \quad \ell \in R_+$$

$$|\top_q(\ell)A^{\wp}| \leq \frac{qC_{\wp}\Gamma(2-\wp)}{\Gamma(1+q(1-\wp))}\ell^{-\wp q}, \quad \ell > 0$$

Now assume that topology of E is the family of semi norms $p_n : m \in M$. E is a Fréchet space.

Definition 2.2. Let $k_{n=1}^{\infty}$ be a series in $[1,2)$. The map $Z : E \to E$ is called to be a compacted function if

$$p_n(\mho(a) - \mho(b)) \leq k_n p_n(a-b) \quad \forall a, b \in E$$

These results are used to show that our main conclusion.

Lemma 2.3. *Assume two operators $Z, C :\to G$. Let underlaying conditions are held*

- *$\mho$ is L_m -contraction, for every $m \in M$,*
- *$\perp$ is throughout continuous and*

$$\lim_{p_n(a)\to\infty} \frac{p_n(\perp(a))}{p_n(a)} = 0, \forall m \in M.$$

Then $\mho + \perp$ has a fixed point in F.

3 Nomenclature

The functional spaces: In this section, for $\xi \geq 0$, we use the representations

$$N_{\xi} := m \in M : m > \xi$$

$$R_{\xi} := [\xi, +\infty)$$

and the following function spaces

(i) $F_{\xi} = F([-s,\xi], F)$ is the metric space of each permenent $\varphi : [-s,\xi] \to F$ of the mode

$$||\varphi||_{\xi} = \sup|\varphi(\ell)| : \ell \in [-s;\xi]$$

(ii) $E_\infty = E([-s,\infty);E)$ is the topological space endowed with the group of semi norms $||.||_{n=1}^{\infty}$.

(iii) $X_{\xi,m} = E([\xi,m];E)$ is the Banach space of all continuous $a:[\xi,m]\to E$ with the norm

$$p_{\xi,m}(a) = \sup_{\ell\in[\xi,m]} |a(\ell)|; a \in X_{\xi,m}$$

$$\varphi(a,b) = \Xi_{n\in N_\xi}^{\infty} 2^{-n} \frac{||a-b||_n}{1+||a-b||_n}, \quad a,b \in E_\infty$$

and

$$d(a,b) = \Xi_{n\in N_\xi}^{\infty} 2^{-n} \frac{p_{\xi,n}(a-b)}{1+p_{\xi,n}(a-b)}, \quad a,b \in X_\xi$$

Let $\xi \geq 0$ and $\varphi \in E_\xi$.

(v) For $a \in X_{\xi,n}$(resp. $a \in X_\xi$), we put

$$a^{\varphi(\ell)} = \begin{cases} a(\ell)+\varphi(\xi)-a(\xi), & \ell\in[\xi,m] \\ \varphi^{(\ell)}, & \ell\in[-s,\xi] \end{cases}$$

or resp.

$$a^{\varphi(\ell)} = \begin{cases} a(\ell)+\varphi(\xi)-a(\xi), & \ell\geq\xi, \\ \varphi^{(\ell)}, & \ell\in[-s,\xi] \end{cases}$$

Then it is truth that $a^\varphi \in E_n$.

(vi) For $t \in r_+$ and $a \in X_{\xi,n}$ we shall represent ξ^φ the map

$$(\ell,a)\mapsto \xi^\varphi(\ell,a) = (\ell, a^\varphi(\ell), a_\ell^\varphi).$$

(vii) At the end, if $\theta \in E_\infty$ then, for every $\ell \in [0,\xi]$(resp. $\ell \in [1;2)$), we will represent by θ_ℓ the E_0 -operator stated by

$$\theta_\ell(\kappa) = \theta(\ell+\kappa), \kappa\in[-r,0]:$$

Definition 3.1.

$$q_m(W(a)-W(b)) \leq l_m q_m(a-b),$$

for all $a,b \in E$. The underlaying results are very useful to prove our main results

4 Hypothesis

To studying the topological structure of mild solution for neutral evolution equations we take the underlaying hypothesis

(h1) A produces an analytic semigroup $\top(\ell)_{\ell\geq 0}$ that is $0\in\rho(B)$ and $\top(\ell)$ is correct for every $\ell>0$.

(h2) The statement $g: E\times\varepsilon\times E\to C_0$ holds the underlaying properties

(i) The map $\ell \mapsto \zeta(\ell, \wp, \nu)$ is measurable, for all $(\wp, \nu) \in E \times C_0$.

(ii) the map $(\wp, \nu) \mapsto \zeta(\ell, \wp, \nu)$ is continuous for a.e. $\ell \in \varepsilon$.

(iii) $\exists$ a constant $q_1 \in [0, q)$ that is for every $D > 0$, there is a positive statement $r_D \in L^{\frac{1}{q_1}}(\varepsilon)$ and if $(\wp, \nu) \in F \times D_0$ with $||(\wp, \nu)|| \leq D$ then $|\zeta(\ell, \wp, \nu)| \leq r_D(\ell)$ for a.e. $\ell \in \varepsilon$.

(iv) for any bounded subset I of E such that $||(\chi b, b)|| \to \infty \frac{|\zeta(\ell,\wp,\nu)|}{||(\wp,\nu)||} = 0$, uniformly in $\ell \in J$.

(h3) $g : \varepsilon \times E \times D_0 \to F$ is compact. Further, $\exists$ a constant $\nu \in (0, 1)$ and a series $Gn > 0 : m \in M$ with

$$G_n\bigg((N+1) \parallel B^{-\nu} \parallel_{L(E)} + \frac{n^{\nu q}\Gamma(1+\nu)D_{1-\nu}}{} \nu\Gamma(1+\nu q)\bigg) < \frac{1}{4}$$

such that $H \in D(A^\nu)$and, for any $(\wp, \nu)$, $(\alpha', \beta') \in E \times C_0$, the function $A^\nu h(., \wp, \nu)$ is measurable and $A^\nu h(\ell, ., .)$ satisfies the Lipschitz condition

$$|A^\nu h(\ell, \wp, \nu) - A^\nu h(\ell, \wp', \nu')| \leq H_n(|\wp - \wp'| + \parallel \nu - \nu' \parallel_0), a.e.\ell \in [0; n]$$

And the operators $S_{q(\ell)_{\ell \geq 0}}$ and $\top_{q(\ell)_{\geq 0}}$ are given by

$$\top_q(\ell)\wp = q \int_0^\infty \theta\psi_q(\theta)\top(\theta)(\ell^q)\wp d\theta$$

$$S_q(\ell)\wp = \int_0^\infty \psi_q(\theta)\top(\theta)(\ell^q)\wp d\theta$$

with $\phi_\wp$ is a probability density function which is defined on this interval $(0, \infty)$ as

$$\phi_q(\vartheta) = \frac{1}{q}\vartheta^{-1-\frac{1}{q}}\psi_q(\theta^{-\frac{1}{q}}) \leq 0,$$

$$\psi_q(\vartheta) = \frac{1}{\pi}\Sigma_{n=1}^\infty(-1)^{n-1}\vartheta^{-nq-1}\frac{\Gamma(nq+1)}{n}\sin(n\pi q)$$

The underlaying results of R_q and $\top_q$ are used in given mild solution.

5 Mild Solution

$$\begin{cases} {}^C D^\alpha \Big[a(\ell) - h(\ell, a(\ell), D^\beta a(\ell))\Big] = Aa(\ell) + f(\ell, a(\ell), D^\beta a(\ell)), \ell > 0 \\ a(0) = a_0 \quad a'(0) = a_1, \ell \in [-r, 0], \\ 1 < \alpha < 2, \quad 0 < \beta < 1 \end{cases}$$

Lemma 5.1. *If*

$$\begin{aligned} a(\ell) &= \chi b_0 + a_1\ell + h(\ell, a(\ell)) - h(0, a_0) - h'(0, \chi b_0)\ell \\ &\quad + \frac{1}{\Gamma(\wp)}(\ell - \kappa)^{\wp-1}a(\kappa)d\kappa + \frac{1}{\Gamma(\wp)}(\ell - \kappa)^{\wp-1}f(\kappa, a(\kappa), a'(\kappa)) \end{aligned}$$

holds then we have

$$\begin{aligned} v(\lambda) &= S_q(\ell)[\chi b_0 + a_1\ell + h(\ell, a(\ell)) - h(0, a_0) - h'(0, a_0)\ell] \\ &+ \int_0^t T_q(\ell-\kappa)(\ell-\kappa)^{q-1} Ah(\kappa, a(\kappa), a'(\kappa))d\kappa \\ &+ \int_0^t T_q(\ell-\kappa)(\ell-\kappa)^{q-1} f(\kappa, a(\kappa), a'(\kappa))d\kappa \end{aligned}$$

Proof.

$$^C D^{\wp}\Big[\chi b(\ell) - h(\ell, a(\ell))\Big] = Aa(\ell) + f(\ell, a(\ell), D^{\nu}a(\ell))$$

Integrate on both sides

$$\begin{aligned} & a(\ell) - \chi b_0 - a_1\ell - h(\ell, a(\ell)) + h(0, a_0) + h'(0, a_0)\ell \\ = & \frac{A}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1} a(\kappa)d\kappa + \frac{1}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1} f(\kappa, \chi b(\kappa), a'(\kappa)) \end{aligned}$$

$$\begin{aligned} a(\ell) &= \chi b_0 + a_1\ell + h(\ell, a(\ell)) - h(0, \chi b_0) - h'(0, a_0)\ell \\ &+ \frac{A}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1} a(\kappa)d\kappa + \frac{1}{\Gamma(\wp)}\int_0^t (\ell-\kappa)^{\wp-1} f(\kappa, \chi b(\kappa), a'(\kappa)) \\ \chi b(\ell) &= a_0 + a_1\ell + h(\ell, \chi b(\ell)) - h(0, a_0) - h'(0, \chi b_0)\ell \\ &+ \frac{1}{\Gamma(\wp)}(\ell-\kappa)^{\wp-1} a(\kappa)d\kappa + \frac{1}{\Gamma(\wp)}(\ell-\kappa)^{\wp-1} f(\kappa, a(\kappa), \chi b'(\kappa)) \end{aligned}$$

$$\begin{aligned} \pounds a(\ell) &= \pounds[a_0] + \pounds[a_1\ell] + \pounds h(\ell, \chi b(\ell)) - \pounds h(0, a_0) \\ &- \pounds h'(0, a_0)\ell + \pounds\frac{A}{\Gamma(\wp)}(\ell-\kappa)^{\wp-1} h(\kappa, a(\kappa), a'(\kappa))d\kappa \\ &+ \pounds\frac{1}{\Gamma(\wp)}(\ell-\kappa)^{\wp-1} f(\kappa, \chi b(\kappa), a'(\kappa))d\kappa \end{aligned}$$

$$\begin{aligned} \int_0^\infty e^-(\lambda)(\kappa)a(\kappa)d\kappa &= a_0 + h(\ell, a(\ell)) - h(0, \chi b_0) + \frac{a_1}{\lambda^2} - \frac{h'(0, a_0)}{\lambda^2} \\ &+ \frac{A}{\lambda^q}\int_0^\infty e^{-(\lambda)(\kappa, a(\kappa), a'(\kappa))} h(\kappa, a(\kappa), a'(\kappa))d\kappa \\ &+ \frac{1}{\lambda^q}\int_0^\infty e^{-(\lambda)(\kappa, aB(\kappa), a'(\kappa))} f(\kappa, a(\kappa), a'(\kappa))d\kappa \end{aligned}$$

Let

$$\begin{aligned} v(\lambda) &= \int_0^\infty e^{-(\lambda)(\kappa, \chi b(\kappa), a'(\kappa))} h(\kappa, a(\kappa), a'(\kappa))d\kappa \\ u(\lambda) &= \int_0^\infty e^{-(\lambda)(\kappa, aB(\kappa), a'(\kappa))} f(\kappa, a(\kappa), a'(\kappa))d\kappa \end{aligned}$$

$$\begin{aligned}
v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) + \frac{a_1}{\lambda^2} - \frac{h'(0, \chi b_0)}{\lambda^2} + \frac{A}{\lambda^q} v(\lambda) + \frac{u(\lambda)}{\lambda^q} \\
v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) + \frac{a_1 - h'(0, a_0)}{\lambda^2} + \frac{Av(\lambda) + u(\lambda)}{\lambda^q} \\
v(\lambda) &= aY_0 + h(\ell, a(\ell)) - h(0, a_0) + a_1[\lambda^2 - A]^{-1} - h'(0, a_0)[\lambda^2 - A]^{-1} \\
&\quad + Av(\lambda)[\lambda^q - A]^{-1} + u(\lambda)[\lambda^q - A]^{-1}
\end{aligned}$$

Put

$$(\lambda^q - A)^{-1} = \int_0^\infty e^{(-\lambda)^q(\kappa)} Q(\kappa) d\kappa \quad (\lambda^2 - A)^{-1} = \int_0^\infty e^{(-\lambda)^q(\kappa)} Q(\kappa) d\kappa$$

$$\begin{aligned}
v(\lambda) &= \chi b_0 + h(\ell, a(\ell)) - h(0, a_0) + a_1 \int_0^\infty e^{(-\lambda^q)(\kappa)} Q(\kappa) d\kappa \\
&\quad - h'(0, a_0) \int_0^\infty e^{(-\lambda^q)(\kappa)} Q(\kappa) d\kappa \\
&\quad + Av(\lambda) \int_0^\infty e^{-(\lambda^q)(\kappa)} Q(\kappa) d\kappa + u(\lambda) \int_0^\infty e^{-(\lambda^q)(\kappa)} Q(\kappa) d\kappa \\
\kappa &= \ell^q \\
d\kappa &= q\ell^{q-1} d\ell \\
v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) + a_1 \int_0^\infty e^{-(\lambda^q)(\ell^q)} Q(\ell^q) q\ell^{q-1} d\ell \\
&\quad - h'(0, a_0) \int_0^\infty e^{-(\lambda^q)(\ell^q)} Q(\ell^q) q\ell^{q-1} d\ell \\
&\quad + Av(\lambda) \int_0^\infty e^{-(\lambda^q)(t^q)} Q(\ell^q) q\ell^{q-1} d\ell + u(\lambda) \int_0^\infty e^{-(\lambda^q)(\ell^q)} Q(\ell^q) q\ell^{q-1} d\ell
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) + a_1 \int_0^\infty e^{-(\lambda\ell)^q} Q(\ell^q) q\ell^{q-1} d\ell \\
&\quad - h'(0, a_0) \int_0^\infty e^{-(\lambda\ell)^q} Q(\ell^q) q\ell^{q-1} d\ell \\
&\quad + Av(\lambda) \int_0^\infty e^{-(\lambda\ell)^q} Q(\ell^q) q\ell^{q-1} d\ell + u(\lambda) \int_0^\infty e^{-(\lambda\ell)^q} Q(\ell^\wp) q\ell^{q-1} d\ell
\end{aligned}$$

put

$$e^{-(\lambda\ell)^q} = \int_0^\infty e^{-(\lambda)(\ell)(\theta)} \psi_q(\theta) d\theta$$

$$\begin{aligned}
v(\lambda) &= aB_0 + h(\ell, a(\ell)) - h(0, \chi b_0) + a_1 \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)(\theta)} \psi_q(\theta) Q(\ell^q) q\ell^{q-1} d\ell d\theta \\
&\quad - h'(0, a_0) \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)(\theta)} \psi_q(\theta) Q(\ell^q) q\ell^{q-1} d\ell d\theta \\
&\quad + Av(\lambda) \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)(\theta)} \psi_q(\theta) Q(\ell^q) q\ell^{q-1} d\ell d\theta \\
&\quad + u(\lambda) \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell)(\theta)} \psi_q(\theta) Q(\ell^q) q\ell^{q-1} d\ell d\theta
\end{aligned}$$

put

$$\begin{aligned}
\ell &= \frac{\ell'}{\theta} \\
d\ell &= \frac{d\ell'}{\theta} \\
v(\lambda) &= aB_0 + h(\ell, a(\ell)) - h(0, a_0) + aB_1 \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} \frac{d\ell'}{\theta} d\theta \\
&-h'(0, a_0) \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} \frac{d\ell'}{\theta} d\theta \\
&+Av(\lambda) \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} \frac{d\ell'}{\theta} d\theta \\
&+u(\lambda) \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} \frac{d\ell'}{\theta} d\theta
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= B_0 + h(\ell, a(\ell)) - h(0, \chi b_0) + \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} a_1 \frac{d\ell'}{\theta} d\theta \\
&-h'(0, a_0) \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} h'(0, a_0) \frac{d\ell'}{\theta} d\theta \\
&+ \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} Av(\lambda) \frac{d\ell'}{\theta} d\theta \\
&+ \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} u(\lambda) \frac{d\ell'}{\theta} d\theta
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= aB_0 + h(\ell, a(\ell)) - h(0, \chi b_0) + \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\
&- \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} h'(0, a_0) d\ell' d\theta \\
&+ \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} Av(\lambda) d\ell' d\theta \\
&+ \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^{q-1}} u(\lambda) d\ell' d\theta
\end{aligned}$$

put the values of $v(\lambda)$, $u(\lambda)$

$$\begin{aligned}
v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, \chi b_0) + \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 dt' d\theta \\
&- \int_0^\infty \int_0^\infty e^{-(\lambda\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\
&+ \int_0^\infty \int_0^\infty e^{-(\lambda)(\ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \\
&\frac{\ell'^{q-1}}{\theta^q} A \int_0^\infty e^{-(\lambda)(\kappa, a(\kappa), a'(\kappa))} h(\kappa, a(\kappa), a'(\kappa)) d\kappa d\ell' d\theta
\end{aligned}$$

$$+\int_0^\infty\int_0^\infty e^{-(\lambda)(\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q$$
$$\frac{\ell'^{q-1}}{\theta^q}\int_0^\infty e^{-(\lambda)(\kappa,a(\kappa),a'(\kappa))}f(\kappa,a(\kappa),a'(\kappa))d\kappa d\ell' d\theta$$

$$\begin{aligned}
v(\lambda) &= a_0+h(\ell,a(\ell))-h(0,a_0)\\
&+a_1\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&-h'(0,a_0)\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&+\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda\ell'+\kappa)}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}Ah(\kappa,a(\kappa),a'(\kappa))d\kappa d\ell' d\theta\\
&+\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda\ell'+\kappa)}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}f(\kappa,a(\kappa),aY'(\kappa))d\kappa d\ell' d\theta
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= a_0+h(\ell,aB(\ell))-h(0,a_0)+a_1\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 dt' d\theta\\
&-h'(0,a_0)\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}aB_1 d\ell' d\theta\\
&+\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda\ell)}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}Ah(\kappa,a(\kappa),a'(\kappa))d\kappa d\ell' d\theta\\
&+\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\lambda\ell)}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}f(\kappa,a(\kappa),a'(\kappa))d\kappa d\ell' d\theta
\end{aligned}$$

$$\begin{aligned}
v(\lambda) &= a_0+h(\ell,aB(\ell))-h(0,\chi b_0)+aB_1\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&-h'(0,a_0)\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&+\int_0^\infty e^{-(\lambda\ell)}\Bigg[\int_0^\infty\int_0^\infty\psi_q(\theta)\frac{Q(t' q)}{\theta^q}q\frac{\ell'^{q-1}}{\theta^q}Ah(\kappa,a(\kappa),a'(\kappa))d\kappa d\theta\\
&+\int_0^\infty\int_0^\infty\psi_q(\theta)\frac{Q(\ell' q)}{\theta^q}q\frac{\ell'^{q-1}}{\theta^q}f(\kappa,a(\kappa),\chi b'(\kappa))d\kappa d\theta\Bigg]d\ell
\end{aligned}$$

Now we invert laplace transform

$$\begin{aligned}
v(\lambda) &= aB_0+h(\ell,a(\ell))-h(0,a_0)+\chi b_1\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&-h'(0,a_0)\int_0^\infty\int_0^\infty e^{-(\lambda\ell')}\psi_q(\theta)\frac{Q(\ell' q)}{(\theta^q)}q\frac{\ell'^{q-1}}{\theta^q}a_1 d\ell' d\theta\\
&+\Bigg[\int_0^\infty\int_0^\infty\psi_q(\theta)\frac{Q(\ell' q)}{\theta^q}q\frac{\ell'^{q-1}}{\theta^q}Ah(\kappa,a(\kappa),a'(\kappa))d\kappa d\theta\\
&+\int_0^\infty\int_0^\infty\psi_q(\theta)\frac{Q(\ell' q)}{\theta^q}q\frac{\ell'^{q-1}}{\theta^q}f(\kappa,a(\kappa),a'(\kappa))d\kappa d\theta\Bigg]
\end{aligned}$$

put $\theta^q = \frac{1}{\theta}$

$$\begin{aligned} v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, aY_0) + a_1 \int_0^\infty \int_0^\infty e^{-(\lambda \ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\ &- h'(0, a_0) \int_0^\infty \int_0^\infty e^{-(\lambda \ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\ &+ \Big[\int_0^\infty \int_0^\infty \psi_q(\theta) Q(\theta)(\ell' q) q(\ell' q - 1) \theta A h(\kappa, a(\kappa), a'(\kappa)) d\kappa d\theta \\ &+ \int_0^\infty \int_0^\infty \psi_q(\theta) Q(\theta)(\ell' q) q(\ell' q - 1) \theta f(\kappa, a(\kappa), a'(\kappa)) d\kappa d\theta \Big] \end{aligned}$$

put

$$\ell' = \ell - \kappa$$

$$\begin{aligned} v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) + \chi b_1 \int_0^\infty \int_0^\infty e^{-(\lambda \ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\ &- h'(0, \chi y_0) \int_0^\infty \int_0^\infty e^{-(\lambda \ell')} \psi_q(\theta) \frac{Q(\ell' q)}{(\theta^q)} q \frac{\ell'^{q-1}}{\theta^q} a_1 d\ell' d\theta \\ &+ \Big[\int_0^\infty \int_0^\infty \psi_q(\theta) Q(\theta)(\ell - \kappa)^q q(\ell - \kappa)^{q-1} \theta A h(\kappa, a(\kappa), a'(\kappa)) d\kappa d\theta \\ &+ \int_0^\infty \int_0^\infty \psi_q(\theta) Q(\theta)(\ell - \kappa)^q q(\ell - \kappa)^{q-1} \theta f(\kappa, aB(\kappa), \chi b'(\kappa)) d\kappa d\theta \Big] \end{aligned}$$

put

$$T_q(\ell) = \int_0^\infty \psi_q(\theta) Q(\theta)(\ell^q) q(\ell^{q-1}) \theta d\theta$$

$$\begin{aligned} v(\lambda) &= a_0 + h(\ell, a(\ell)) - h(0, a_0) \\ &+ \int_0^\infty (\ell - \kappa)^{q-1} a_1, a'(\kappa)) d\kappa \Big[\int_0^\infty \psi_q(\theta) Q((\theta)((\ell - \kappa)^q)) q\theta d\theta \Big] \\ &- \int_0^\infty (\ell - \kappa)^{q-1} h'(0, a_0), a'(\kappa)) d\kappa \Big[\int_0^\infty \psi_q(\theta) Q((\theta)((\ell - \kappa)^q)) q\theta d\theta \Big] \\ &+ \int_0^\infty (\ell - \kappa)^{q-1} A h(\kappa, a(\kappa), a'(\kappa)) d\kappa \Big[\int_0^\infty \psi_q(\theta) Q((\theta)((\ell - \kappa)^q)) q\theta d\theta \Big] \\ &+ \int_0^\infty ((t - \kappa)^{q-1}) f(\kappa, \chi b(\kappa), a'(\kappa)) d\kappa \Big[\int_0^\infty \psi_q(\theta) Q(\theta)((\ell - \kappa)^q) q\theta d\theta \Big] \end{aligned}$$

$$\begin{aligned} v(\lambda) &= aY_0 + h(\ell, a(\ell)) - h(0, a_0) + \in \ell_0^t T_q(\ell - \kappa)(\ell - \kappa)^{q-1} a_1, a'(\kappa)) d\kappa \\ &- \int_0^t \top_q(\ell - \kappa)(\ell - \kappa)^{q-1} h'(0, a_0), a'(\kappa)) d\kappa \\ &+ \int_0^t \top_q(\ell - \kappa)(\ell - \kappa)^{q-1} A h(\kappa, a(\kappa), a'(\kappa)) d\kappa \\ &+ \int_0^t \top_q(\ell - \kappa)(\ell - \kappa)^{q-1} f(\kappa, a(\kappa), a'(\kappa)) d\kappa \end{aligned}$$

$$\begin{aligned} v(\lambda) &= S_q(\ell)[a_0 + h(\ell, a(\ell)) - h(0, a_0) + \int_0^t \top_q(\ell-\kappa)(\ell-\kappa)^{q-1} a_1, a'(\kappa))d\kappa \\ &- \int_0^t \top_q(t-\kappa)(\ell-\kappa)^{q-1} h'(0, a_0), a'(\kappa))d\kappa] \end{aligned}$$

$$\begin{aligned} &+ \int_0^t \top_q(\ell-\kappa)(\ell-\kappa)^{q-1} A h(\kappa, a(\kappa), a'(\kappa))d\kappa \\ &+ \int_0^t \top_q(\ell-\kappa)(\ell-\kappa)^{q-1} f(\kappa, a(\kappa), a'(\kappa))d\kappa \end{aligned}$$

This is a mild solution of fractional neutral evolution equation. □

6 Example

Example 6.1. The operator $_m$ is completely continuous relation

$$lim_{p_{0,m}(\wp)\to\infty} \frac{p_{0,m}(\mho_m(\wp))}{p_{0,m}(\wp)} = 0.$$

Proof. We divide this example into three steps.

Step 1. $\mho_m$ is compact. Indeed, let $(\wp_k)$ is a series in $X_{0,m}$ meet to $\wp \in X_{0,m}$. Put

$$H = \{\wp_o : o \in N\} \cup \{\wp\}$$

Since $\partial^{\psi}([0, m] \times G)$ is connect there is a positive function $s_G \in L^{\frac{1}{q_1}}(\varepsilon)$ such that

$$|f(\partial^{\psi}(\kappa, b))| \le r_G(\kappa)$$

$\forall\ b \in G$ and for a.e. $\kappa \in [0, m]$. This implies

$$|f(\partial^{\psi}(\kappa, \wp_k)) - f((\partial^{\psi}(\kappa, \wp))| \le 2r_G(\kappa),$$

$\forall\ l \in M$ and for a.e. $\kappa \in [0, m]$. In the contrary, it taken from the compactness of ∂^{ψ} and the preassumtion that $g(\eta(\kappa, \eta\wp_k)) - g(\eta(\kappa, a))$ meets to 0, for atmost each $\kappa \in [0, m]$. Hence, by this theorem

$$lim_{k\to\infty} \int_0^n |\zeta(\partial^{\psi}(\kappa, \wp_k)) - \zeta(\partial^{\psi}(\kappa, \wp))|^{\frac{1}{q_1}} d\kappa = 0.$$

From Hölder inequality we get

$$\begin{aligned}
\Big|\mho_m\wp_k(\ell)-\mho_m\wp(\ell)\Big| &= \Big|\int_0^t(\ell-\kappa)^{q-1}\top_q(\ell-\kappa)[f(\partial^\psi(\kappa,\wp_k)-f(\partial^\psi(\kappa,\wp))]d\kappa\Big|\\
&\le \frac{qM}{\Gamma(1+q)}\int_0^t(\ell-\kappa)^{q-1}[\zeta(\partial^\psi(\kappa,\wp_k)-f(\partial^\psi(\kappa,\wp))]d\kappa\\
&\le \frac{qM}{\Gamma(1+q)}\left(\int_0^t(\ell-\kappa)^{\frac{q-1}{1-q_1}}d\kappa\right)^{1-q_1}\\
&\times\left(\int_0^t|\zeta(\partial^\psi(\kappa,\wp_k))-f(\partial^\psi(\kappa,\wp))|^{\frac{1}{q_1}}(\kappa)d\kappa\right)^{q_1}
\end{aligned}$$

$$\le \frac{qMm^{q-q_1}}{\Gamma(1+q)}(\frac{1-q_1}{q-q_1})^{1-q_1}\parallel\zeta(\partial^\psi(.,\wp_k))-\zeta(\partial^\psi(.,\wp))\parallel_{L^{\frac{1}{q_1}}}(0,m)$$

$\forall\ \ell\in[0,m]$. Here we apply the underlaying suggestion

$$\left(\int_0^t(\ell-\kappa)^{\frac{q-1}{1-q_1}}d\kappa\right)=\ell^{q-q_1}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\le m^{q-q_1}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}$$

Hence $p_{0,m}(\mho_m a_k-\mho_m a)$ meets to 0 when $L\to\infty$. Therefore, $\mho_m$ is continuous.

Step 2. $\mho_m$ is connect. Let R be a bounded subset of Z_{0m}. when we use this Lemma the set $\partial^\psi([0;n]\times R)$ is bounded. Hence there is a positive statement $s_Q\in L^{\frac{1}{q_1}}(\varepsilon)$ such as

$$|\zeta(\partial^\psi(\ell,\wp))|\le r_Q(\ell)$$

for all $a\in Q$ and for a.e. $\ell\in[0,m]$. For $a\in Q$ and $0\le\ell_1<\ell_2\le m$, we have

$$\begin{aligned}
\Big|B_m\wp_(\ell_2)-\mho_m\wp(\ell_1)\Big| &= \Big|\int_0^t(\ell_2-\kappa)^{q-1}\top_q(\ell_2-\kappa)\zeta(\partial^\psi(\kappa,\wp))d\kappa\\
&-\int_0^t(\ell_1-\kappa)^{q-1}\top_q(\ell_1-\kappa)\zeta(\partial^\psi(\kappa,\wp))d\kappa\Big|\\
&\le I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \Big|\int_0^t(\ell_2-\kappa)^{q-1}\top_q(\ell_2-\kappa)\zeta(\partial^\psi(\kappa,\wp))d\kappa\Big|\\
I_2 &= \Big|\int_0^t[(\ell_2-\kappa)^{q-1}-(\ell_1-\kappa)^{q-1}]\top_q(\ell_2-\kappa)\zeta(\partial^\psi(\kappa,\wp))d\kappa\Big|\\
I_3 &= \Big|\int_0^t(\ell_1-\kappa)^{q-1}[\top_q(\ell_2-\kappa)-\top_q(\ell_1-\kappa)]\zeta(\partial^\psi(\kappa,\wp))d\kappa\Big|
\end{aligned}$$

Estimate I_1. By using Hölder's inequality we have

$$I_1\le\frac{qM}{\Gamma(1+q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\parallel r_Q\parallel_{L^{\frac{1}{q_1}}_{(0,m)}}(\ell_2-\ell_1)^{q-q_1}:$$

Estimate I_2. We shall use the following inequality

$$(a^{\wp} - b^{\wp})^{\gamma} \leq a^{\wp\gamma} - b^{\wp\gamma},$$

$\forall\ 0 < c < d$, $\wp < 0$, $\gamma > 1$. For justifying this equality we write that the function

$$o(\ell) = (\ell^{\wp} - 1)^{\gamma} - \ell^{\wp\gamma} + 1$$

is decreasing on $(0, 1]$

$$o'(\ell) = \wp\gamma[(\ell^{\wp} - 1)^{\gamma-1}\ell^{\wp-1} - \ell^{\wp\gamma-1}] \geq 0; \quad \ell \in (0, 1].$$

Hence $o(\ell) \leq o(1) = 0$ which shows that

$$(\ell^{\wp} - 1)^{\gamma} \leq \ell^{\wp\gamma} - 1.$$

Let $\ell = \frac{a}{b}$ we gain the required inequality. By using the Hölder's inequality we get

$$\begin{aligned} I_2 &\leq \frac{qM}{\Gamma(1+q)} \parallel r_Q \parallel_{L^{\frac{1}{q_1}}_{(0,m)}} \left(\int_0^t \left[(\ell_2 - \kappa)^{q-1} - (\ell_1 - \kappa)^{q-1} \right]^{\frac{1}{1-q_1}} d\kappa \right)^{1-q_1} \\ &\leq \frac{qM}{\Gamma(1+q)} \left(\frac{1-q_1}{q-q_1} \right)^{1-q_1} \parallel r_Q \parallel_{L^{\frac{1}{q_1}}_{(0,m)}} \left(\ell_1^{\frac{q-q_1}{1-q_1}} - \ell_2^{\frac{q-q_1}{1-q_1}} \right. \\ &\quad \left. + (\ell_2 - \ell_1)^{\frac{q-q_1}{1-q_1}} \right)^{1-q_1} \\ &\leq \frac{qM}{\Gamma(1+q)} \left(\frac{1-q_1}{q-q_1} \right)^{1-q_1} \parallel r_Q \parallel_{L^{\frac{1}{q_1}}_{(0,m)}} (\ell_2 - \ell_1)^{q-q_1}. \end{aligned}$$

Estimate I_3. Without loss of generality we let that $\ell_1 > 0$. For $\epsilon > 0$ littel enough

$$\begin{aligned} I_3 &\leq \int_0^{\ell_1-\epsilon} (\ell_1 - \kappa)^{q-1} |\top_q(\ell_2 - \kappa)\zeta(\partial^{\psi}(\kappa, \wp)) - \top_q(\ell_1 - \kappa)\zeta(\partial^{\psi}(\kappa, \wp))| d\kappa \\ &\quad + \int_{\ell_1-\epsilon}^{\ell_1} (\ell_1 - \kappa)^{q-1} |\top_q(\ell_2 - \kappa)\zeta(\partial^{\psi}(\kappa, \wp)) - \top_q(\ell_1 - \kappa)\zeta(\partial^{\psi}(\kappa, a\wp))| d\kappa \\ &\leq \left(\frac{1-q_1}{q-q_1} \right)^{1-q_1} \parallel r_Q \parallel_{L^{\frac{1}{q_1}}_{[0,m]}} \left[\frac{2qM\epsilon^{1-q_1}}{\Gamma(1+q)} \right. \\ &\quad \left. + \left(\ell_1^{\frac{q-q_1}{1-q_1}} - \epsilon^{\frac{q-q_1}{1-q_1}} \right)^{1-q_1} \sup_{\kappa \in [0, \ell_1 - \epsilon]} \parallel \top_q(\ell_2 - \kappa) - \top_q(\ell_1 - \kappa) \parallel_{L(E)} \right] \end{aligned}$$

Since $\top(\ell)$ is compact, it follows that $\top_q(\ell)(\ell > 0)$ is compact in ℓ in the variable metric space. Hence I_3 meets to zero dependently of $\wp \in Q$ as $\ell_2 - \ell_1 \to 0$ and $\epsilon \to 0$. By adding the upper results we can results that $\{\mho_n \chi b\wp : \wp \in Q\}$ is equicontinuous. For each $\ell \in [0, m]$ we take

$$K(t) = \{\mho_m \wp(\ell) : \wp \in R\}$$

It is important to prove the relative compactness in F of $K(\ell)$. For these $\epsilon \in (0, \ell)$ and $\delta > 0$,

$$\mho_m^{\epsilon}\wp(\ell) = \int_0^{\ell-\epsilon}\int_{\delta}^{\infty} q\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top[(\ell-\kappa)^q\theta]\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa$$

for $a \in R$ and $\ell \in [0, m]$. It is easy to know that

$$\mho_m^{\epsilon}\wp(\ell) = \top(\epsilon^q\delta)q\int_0^{t-\epsilon}\int_{\delta}^{\infty} \theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top[(\ell-\kappa)^q\theta - \epsilon^q\delta]\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa$$

By using the results on g and the boundedness of R, it follows that the set

$$q\int_0^{t-\epsilon}\int_{\delta}^{\infty} \theta(\ell-s)^{q-1}\phi_q(\theta)\top[(\ell-\kappa)^q\theta - \epsilon^q\delta]\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa : \wp \in Q$$

is bounded in F. Then, from the connectedness of $\top(\epsilon^q\delta)$, we can obtain that

$$L^{\epsilon(\ell)} = \mho_m^{\epsilon}\wp(\ell) : \wp \in R$$

is relatively compact in F. On the contrary since

$$\top_q(\ell)b = q\int_0^{\infty}\theta\phi(\theta)\top(\ell^q\theta)bd\theta, \quad b \in F$$

we conclude that, for every$a \in R$,

$$\begin{aligned}
|\mho_m a_k(\ell) - \mho_m\chi b(\ell)| &= q\Big|\int_0^t\int_0^{\delta}\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top((\ell-\kappa)^q\theta)\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa \\
&\quad + \int_0^t\int_{\delta}^{\infty}\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top((\ell-\kappa)^q\theta)\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa \\
&\quad - \int_0^{t-\epsilon}\int_{\delta}^{\infty}\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top[(\ell-\kappa)^q\theta]\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq q\Big|\int_0^t\int_0^{\delta}\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top((\ell-\kappa)^q\theta)\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa\Big| \\
&\quad + q\Big|\int_{t-\epsilon}^t\int_{\delta}^{\infty}\theta(\ell-\kappa)^{q-1}\phi_q(\theta)\top[(\ell-\kappa)^q\theta]\zeta(\partial^{\psi}(\kappa,\wp))d\theta d\kappa\Big| \\
&\leq q(O_1 + O_2)
\end{aligned}$$

By using Hölder's inequality we suggest O_1 and O_2 as follows:

$$\begin{aligned}
O_1 &\leq N\Big(\int_0^t(\ell-\kappa)^{q-1}r_Q(\kappa)d\kappa\Big)\Big(\int_0^{\delta}\theta\phi_q(\theta)d\theta\Big) \\
&\leq N\Big(\int_0^t(\ell-\kappa)^{\frac{q-1}{1-q_1}}d\kappa\Big)^{1-q_1}\Big(\int_0^t r_Q^{\frac{1}{q_1}}(\kappa)d\kappa\Big)^{q_1}\Big(\int_0^{\delta}\theta\phi_q(\theta)d\theta\Big) \\
&\leq N\Big(\frac{1-q_1}{q-q_1}\Big)^{1-q_1} n^{q-q_1}||r_Q||_{L^{\frac{1}{q_1}}_{(0,m)}}\Big(\int_0^{\delta}\theta\phi_q(\theta)d\theta\Big)
\end{aligned}$$

and

$$\begin{aligned} O_2 &\leq M\bigg(\int_{t-\epsilon}^{t}(\ell-\kappa)^{q-1}r_Q(\kappa)d\kappa\bigg)\bigg(\int_{\delta}^{\infty}\theta\phi_q(\theta)d\theta\bigg) \\ &\leq M\bigg(\int_{t-\epsilon}^{t}(\ell-\kappa)^{\frac{q-1}{1-q_1}}d\kappa\bigg)^{1-q_1}\bigg(\int_{t-\epsilon}^{t}r_Q^{\frac{1}{q_1}}(\kappa)d\kappa\bigg)^{q_1}\bigg(\int_{0}^{\delta}\theta\phi_q(\theta)d\theta\bigg) \\ &\leq M\bigg(\frac{1-q_1}{q-q_1}\bigg)^{1-q_1}\epsilon^{q-q_1}\parallel r_Q\parallel_{L^{\frac{1}{q_1}}_{[0,m]}}\bigg(\int_{0}^{\infty}\theta\phi_q(\theta)d\theta\bigg) \end{aligned}$$

Combining above inequalities and noting that

$$\int_0^\infty \theta\phi_q(\theta)d\theta = \frac{1}{\Gamma(q+1)}$$

we obtain

$$|\mho_m a_k(\ell)-\mho_m a(\ell)| \leq M\bigg(\frac{1-q_1}{q-q_1}\bigg)^{1-q_1}\parallel r_Q\parallel_{L^{\frac{1}{q_1}}_{[0,m]}}\bigg[m^{q-q_1}\int_0^\infty\theta\phi_q(\theta)d\theta+\frac{\epsilon^{q-q_1}}{\Gamma(q+1)}\bigg]$$

Hence there are present relatively fractional sets arbitrarily open to the set$L(\ell)$. So $L(\ell)$ is also relatively connect in F.

Step 3. Let $\epsilon > 0$. Its clear that $\exists$ a negitive arbitrary constant E such as

$$|f(\ell,\wp,b)| \leq \frac{\Gamma(1+q)_\epsilon}{4Mm^q}\parallel(\wp,b)\parallel$$

$\forall\ \ell\in[0;n]$ and $\forall\ (\wp,b)\in F\times D_0$ with $\parallel(\wp,b)\parallel > C$. Let $s_D\in L_{\frac{1}{q_1}}(\varepsilon)$ be a nonnegative function satisfying

$$|g(\ell,\wp,b)| \leq s_D(\ell),$$

a.e. $\ell\in\varepsilon$ regarded that $\parallel(\wp,b)\parallel\leq C$. Hence, $\forall\ (\chi,b)\in F\times D_0$,

$$|g(\ell,a,b)| \leq s_D(\ell)+\frac{\Gamma(1+q)_\epsilon}{4Mn^q}\parallel(\wp,b)\parallel$$

a.e. $\ell\in[0,m]$. By using theorem, for $a\in X_{0,m}$ and $\ell\in[0,m]$, we take

$$\begin{aligned} |\mho_n a(\ell)| &\leq \frac{qM}{\Gamma(1+q)}\int_0^t(\ell-\kappa)^{q-1}|\zeta(\partial^\psi(\kappa,\wp))|d\kappa \\ &\leq \frac{qM}{\Gamma(1+q)}\int_0^t(\ell-\kappa)^{q-1}\bigg[r_C(\kappa)+\frac{\Gamma(1+q)_\epsilon}{4Mm^q}(|a^\psi(\kappa)|+||a_\kappa^\psi||_m)\bigg]d\kappa \\ &\leq \frac{qM}{\Gamma(1+q)}\int_0^t(\ell-\kappa)^{q-1}\bigg[r_C(\kappa)+\frac{\Gamma(1+q)_\epsilon}{4Mm^q}(4p_{0,m}(a)+2||\psi||_0)\bigg]d\kappa \\ &\leq \frac{qM}{\Gamma(1+q)}\bigg[\bigg(\frac{1-q_1}{q-q_1}\bigg)^{1-q_1}m^{q-q_1}||r_C||_{L^{\frac{1}{q_1}}_{[0,m]}} \\ &\quad+\frac{\Gamma(1+q)_\epsilon}{4Mm^q}(4p_{0,m}(a)+2||\psi||_0)\bigg] \\ &\leq \epsilon p_{0,m}(a)+\frac{qM}{\Gamma(1+q)}\bigg(\frac{1-q_1}{q-q_1}\bigg)^{1-q_1}m^{q-q_1}||r_C||_{L^{\frac{1}{q_1}}_{[0,m]}}+\frac{\epsilon}{2}\parallel\psi\parallel_0 \end{aligned}$$

So

$$lim_{p_{0,m}(a)\to\infty}\frac{p_{0,m}(\mho_m(a))}{p_{0,m}(\chi,b)}\leq\epsilon.$$

This shows that

$$lim_{p_{0,m}(a)\to\infty}\frac{p_{0,m}(\mho_m(a))}{p_{0,m}(a)}=0$$

because ϵ is constant. The proof of this lemma is complete. □

7 Author's Contributions

All author's contributed equally to the writing of this paper. All author's read and approved the final manuscript.

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