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Approximating solutions of fuzzy stochastic fractional integro-evolution equations with the Averaging Principle: Theory and Applications

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Abstract This paper introduces the averaging principle (AP) as a method for solving fuzzy stochastic fractional integro-evolution equations (FSFIEEs). By making certain assumptions, the solutions of FSFIEEs can be estimated as mean square solutions of averaged fuzzy stochastic systems. This technique simplifies the analysis and comprehension of complex systems that are subject to both randomness and uncertainty.

Keywords: Fuzzy stochastic fractional evolution equations; Mild solution; Existence; uniqueness; Averaging principle; Gronwall inequality.

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1. Introduction

In physics and a variety of other subjects, the averaging approach is a strong instrument for examining the qualitative properties of dynamical systems. This method demonstrates a link between averaged system solutions and standard form solutions (1; 2). However, the AP for FSFIEEs has yet to be investigated in the literature. We make the first attempt to investigate this strategy in this work.

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(3; 4; 5) contain pivotal results on the AP for crisp stochastic differential equations (SDEs). Tan et al. (6) were the first to propose the non-Lipschitz averaging method for stochastic differential delay equations (SDDEs). Mao et al. (7) worked on the AP for SDDEs with jump in 2015, in 2014, Xu (8) worked with fractional Brownian motion and Guo (9) worked with nonlinear terms that fulfill the monotone in 2022. While Luo et al. (10; 11) evaluated the AP for a class of stochastic fractional differential equations (SFDEs) with space-time delays. The AP for Hilfer fractional SDDEs with Poisson jumps was established by Ahmed et al. (12). FSDEs are applied to modern-world systems where the phenomenon is associated with fuzziness and randomness, respectively. The authors gave a definition of the fuzzy stochastic Ito integral in (13; 14), employing a mechanism that permits a crisp Ito stochastic integral to be embedded into a fuzzy space for the construction of a fuzzy random variable (FRV). Abbas et al. (15; 16) solve ODE. Niazi et al.(17), Iqbal et al. (18), Shafqat et al. (19), Alnahdi (20), Khan (21) and Abuasbeh et al. (22; 23; 24) existence and uniqueness of the FFEEs were investigated. Arhrrabi et al. (25) worked on the averging principle for FSDEs by using the following problem:

$$\begin{cases} dx(s) = \mathscr{Q}(\omega,x(\omega)d\omega + \langle \mathscr{H}(\omega,x(\omega))d\mathscr{B}(\omega)\rangle, \\ x(0) = x_0 \in \mathbf{E_m}, \end{cases}$$

where $\mathscr{Q} : \imath \times \mathbf{E_m} \to \mathbf{E_m}, \mathscr{H} : \imath \times \mathbf{E_m} \to \mathbf{R_m}$ and $x_0 : \Omega \to \mathbf{E_m}$ is a FRV. Inspired by the above work, the author worked on the averaging principle for FSFEEs by using the following problem:

$$\begin{cases} {}^{c}D^{\varpi}_{\omega}U(\omega) = AU(\omega) + \mathscr{B}\int_0^{\omega} \mathscr{Q}(\omega,U(h(\omega))d\omega + \langle \mathscr{H}(\omega,U(\omega))d\mathscr{B}(\omega)\rangle, \\ U(\omega) = \eta_0, \\ U'(\omega) = \eta_1 \in \mathbf{E_m}, \end{cases}$$

where $\mathscr{Q} : \imath \times \mathbf{E_m} \to \mathbf{E_m}, \mathscr{H} : \imath \times \mathbf{E_m} \to \mathbf{R_m}$ and $\eta_0, \eta_1 : \Omega \to \mathbf{E_m}$ is a FRV.

The manuscript discusses a mathematical principle called “the averaging principle” (AP) that can be used to solve fuzzy stochastic fractional integro-evolution equations (FSFIEEs). These equations describe systems that evolve over time and are affected by both randomness and uncertainty. The AP involves approximating the solutions of FSFIEEs as mean square solutions of averaged fuzzy stochastic systems. This means that, under certain assumptions, the behavior of the original system can be effectively captured by a simpler system that is easier to analyze and understand. The manuscript likely includes a detailed explanation of the AP and how it can be applied to FSFIEEs. The assumptions required for the AP to be valid are also likely discussed. The paper may also include examples of how the AP has been applied in real-world scenarios. The goal of this study is to apply the Caputo derivative on AP to FSFIEEs. The remaining work is organized as follows. Section 2 provides the basic definitions and properties needed throughout this paper. The average method for FS-FIEEs is examined in Section 3 under some scenarios. The major finding of this work is illustrated with an example in Section 4. Furthermore, Section 5 has the conclusion.

2. Preliminaries

In this section, we provide an introduction to the notation, definitions, and historical context that will be utilized throughout the paper. $\mathscr{K}(\mathbf{R_m})$ is the nonempty family of compact and convex subsets of $\mathbf{R_m}$. The d_H distance in $\mathscr{K}(\mathbf{R_m})$ is defined as

$$d_{\mathscr{H}}(\mathscr{B},\mathscr{D}) : \max\left(\sup_{b\in\mathscr{B}}\inf_{d\in\mathscr{D}}||b-d||, \sup_{d\in\mathscr{D}}\inf_{b\in\mathscr{B}}||b-d||\right),\ \mathscr{B},\mathscr{D}\in\mathscr{K}(\mathbf{R_m}).$$

With respect to $d_{\mathscr{H}}$, it is known that $\mathscr{K}(\mathbf{R_m})$ is a separable complete metric space. Let $\mathbf{E_m}$ be the fuzzy space of $\mathbf{R_m}$, which is the set of functions $V:\mathbf{R_m}\to[0,1]$ in which $[V]^{\alpha}\in\mathscr{K}(\mathbf{R_m}),\forall\in[1,2]$ where

$$V^{\varpi} := \{a\in\mathbf{R_m} : V(a)\geqslant\varpi\},\ for\ \varpi\in[1,2],$$

and

$$[V]^0 := cl\{a\in\mathbf{R_m} : V(\varpi)>0\}.$$

Assume

$$d_\infty(U,V) := \sup_{\varpi\in[0,1]} d_{\mathscr{H}}([a]^{\varpi},[b]^{\varpi})$$

be the metric satisfying the following properties:

(i) $d_\infty(a+c,b+c) = d_\infty(a,b)$,

(ii) $d_\infty(a+b,c+X) \leqslant d_\infty(a,c)+d_\infty(b,X)$,

(iii) $d_\infty(\lambda a,\lambda b) = |\lambda| d_\infty(a,b), \lambda \in \mathbf{R_m}$.

Assume $\langle . \rangle : \mathbf{R_m} \to \mathbf{E_m}$ be an embedding of $\mathbf{R_m}$ into $\mathbf{E_m}$ that is for $r^m \in \mathbf{R_m}$, one has

$$\langle r^m \rangle(a) = \begin{cases} 1, \; if \; a = r^m, \\ 0, \; if \; a \neq r^m. \end{cases}$$

Remark 1. If $V : [0,T] \times \Omega \to \mathbf{R_m}$ is a $\mathbf{R_m}$-valued stochastic process (SP), then $\langle V \rangle : [0,T] \times \Omega \to \mathbf{E_m}$ is a fuzzy SP.
Assume $\{\mathscr{B}(\omega), \omega \in \imath := [0,T]\}$ be a 1D Brownian motion described on (Ω,A,P) which is a complete probability space with a filtration $\{A_\omega\}_{\omega\in[0,T]}$ fulfilling conventional hypotheses.

Definition 1. (26) We mean the FRV $\langle \int_0^\omega V(\nu)d\mathscr{B}(\nu)\rangle$ by fuzzy stochastic *Itô* integral. Let the fuzzy stochastic *Itô* integral $\langle \int_0^\omega V(\nu)d\mathscr{B}(\nu)\rangle$ for every $\omega \in \imath$, which can be construed as shown in:

$$\left\langle \int_0^\omega V(\nu)d\mathscr{B}(\nu) \right\rangle := \left\langle \int_0^T \chi_{[0,\omega]}(\nu)V(\nu)d\mathscr{B}(\nu) \right\rangle.$$

Proposition 1. (26) If $U,V \in \ell^2(\imath \times \Phi, \mathscr{N}; \mathbf{R_m})$, then $\forall \omega \in \imath$, we have

$$d_\infty^2 \left\langle \int_0^\omega U(\nu)d\mathscr{B}(\nu) \right\rangle := \left\langle \int_0^T \chi_{[0,\omega]}(\nu)V(\nu)d\mathscr{B}(\nu) \right\rangle.$$

Proposition 2. (27) If $U,V \in \ell^p(\imath \times \Phi, \mathscr{N}; \mathbf{E_m})$, and $p \geqslant 1$, we have

$$E \sup_{a\in[0,\omega]} d_\infty^p \left(\int_0^a U(\nu)d\nu, \int_0^\omega V(\nu)d\nu \right) \leqslant \omega^{p-1} \int_0^\omega E d_\infty^p(U(\nu),V(\nu))d\nu.$$

3. Main Result

Consider the following FSDEs

$$\begin{cases} {}^cD^{\varpi}_{\omega}U(\omega) = AU(\omega) + \mathscr{B}\int_0^{\omega}\mathscr{Q}(\omega,U(h(\omega))d\omega + \langle\mathscr{H}(\omega,U(\omega))d\mathscr{B}(\omega)\rangle, \\ U(t) = \eta_0, \\ U'(\omega) = \eta_1 \in \mathbf{E_m}, \end{cases} \tag{1}$$

where $\mathscr{Q} : \imath \times \mathbf{E_m} \to \mathbf{E_m}, \mathscr{H} : \imath \times \mathbf{E_m} \to \mathbf{R_m}$ and $\eta_0, \eta_1 : \Omega \to \mathbf{E_m}$ is a FRV. The mild solution of equation 1 is

$$\begin{aligned} U(\omega) &= C_q(\omega)\eta_0 + K_q(\omega)\eta_1 + \frac{1}{\Gamma(\beta)}\int_0^{\omega}(\omega-s)^{q-1}P_q(\omega-s) \\ &\quad \left[B\int_0^{\omega}\mathscr{Q}(\nu,U(h(\nu)))d\nu + \left\langle\mathscr{H}(\nu,U(\nu))d\mathscr{B}(\nu)\right\rangle\right]d\nu. \end{aligned} \tag{2}$$

where $\omega \in [0,b]$. To demonstrate that the solution to 1 exists and is distinct, we apply conditions to the coefficient functions.
$(\mathscr{A}_1)$ There exists a constant $C_1 > 0$ ensures we have $\forall\omega \in \imath$ and $\forall x \in \mathbf{E_m}$:

$$d^2_{\infty}(\mathscr{Q}(\omega,U),\hat{0}) \leqslant C_1^2(1 + d^2_{\infty}(x,\hat{0})$$

and

$$\|\mathscr{H}(\omega,U)\|^2 := d^2_{\infty}(\langle\mathscr{H}(\omega,U)\rangle,\hat{0}) \leqslant C_1^2(1 + d^2_{\infty}(U,\hat{0})).$$

$(\mathscr{A}_2)$ There exists $C_2 > 0$ a constant s.t we have $\forall\omega \in \imath$ and $\forall U \in \mathbf{E_m}$:

$$d^2_{\infty}(\mathscr{Q}(\omega,U),\mathscr{Q}(\omega,V)) \leqslant C_2(1 + d^2_{\infty}(U,V))$$

and

$$\|\mathscr{H}(\omega,U) - \mathscr{H}(\omega,V)\|^2 := d^2_{\infty}(\langle\mathscr{H}(\omega,U)\rangle,\langle\mathscr{H}(\omega,V)\rangle) \leqslant C_2(1 + d^2_{\infty}(U,V).$$

According to the findings of Malinowski and Michta (26), FSFIEEs 1 has a solution that is unique $U(\omega)$ with the initial data η_0 and η_1 under the assumptions $(\mathscr{A}_1)$ and

$(\mathscr{A}_2)$. Take the standard form of an equation 2

$$
\begin{aligned}
U_\varepsilon(\omega) &= C_q(\omega)\eta_0 + K_q(\omega)\eta_1 + \frac{1}{\Gamma(\beta)}\int_0^\omega (\omega-\nu)^{q-1}P_q(\omega-\nu) \\
&\quad \left[\varepsilon\mathscr{B}\int_0^\omega \mathscr{Q}(\nu,U(h(\nu)))d\nu + \sqrt{\varepsilon}\left\langle \mathscr{H}(\nu,U(\nu))d\mathscr{B}(s)\right\rangle\right]d\nu, \quad (3)
\end{aligned}
$$

where the initial value are η_0 and η_1, functions $\mathscr{Q}$ and $\mathscr{H}$ having conditions similar as in 2, and $\varepsilon \in (0,\varepsilon_0)$ and $\varepsilon > 0$ with a fixed value. According to the existence and uniqueness results, Eq. 3 has a unique solution $U_\varepsilon(\omega)$ for each fixed $\varepsilon \in (0,\varepsilon_0)$ and $\omega \in \imath$. To determine whether a simple process can approximate the solution $U_\varepsilon(\omega)$, we make specific assumptions about the coefficients.
Let $\tilde{\mathscr{Q}} : \mathbf{E_m} \to \mathbf{E_m}$ and $\tilde{\mathscr{H}} : \mathbf{E_m} \to \mathbf{R_m}$ be measurable functions satisfy the condition $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$, as well as:
(A_3) For $x \in \mathbf{E_m}$ and $\mathfrak{I}^q \in \imath$, we have

$$
\frac{1}{\mathfrak{I}^q}\int_0^{\mathfrak{I}} d_\infty^2(\mathscr{Q}(\nu,U),\tilde{\mathscr{Q}}(U))d\nu \leqslant \beta_1(T')(1+d_\infty^2(U,\hat{0})
$$

and

$$
\frac{1}{\mathfrak{I}'}\int_0^{T'} \|\mathscr{H}(\nu,U)-\tilde{\mathscr{H}}(U)\|^d\nu \leqslant \beta_2(T')(1+d_\infty^2(U,\hat{0}).
$$

where $\lim\limits_{\mathfrak{I}'\to\infty}\beta_i(T') = 0, i = 1,2.$
We now demonstrate, using the proper preparations, that the solution $U_\varepsilon \to V_\varepsilon$ of the given averaged FSFIEEs.

$$
\begin{aligned}
V_\varepsilon(\omega) &= C_q(\omega)\eta_0 + K_q(\omega)\eta_1 + \frac{1}{\Gamma(\beta)}\int_0^\omega (\omega-\nu)^{q-1}P_q(\omega-\nu) \\
&\quad \left[\varepsilon B\int_0^\omega \tilde{\mathscr{Q}}(\nu,V(h(\nu)))d\nu + \sqrt{\varepsilon}\left\langle \tilde{\mathscr{H}}(\nu,V(\nu))d\mathscr{B}(\nu)\right\rangle\right]d\nu, \quad (4)
\end{aligned}
$$

as $\varepsilon \to 0$. Equations 3 and 4 also have a V_ε solution under equivalent assumptions. The main outcome of this work is the theorem that follows, which examines the relationships between U_ε and V_ε.

Theorem 1. Suppose the statements $(\mathscr{A}_1)-(\mathscr{A}_3)$ are fulfilled. There exists $\varepsilon_1 \in (0,\varepsilon_0]$ for very small numbers Δ, $k>0$ and $\varpi \in (1,2)$ such that we have $\forall \varepsilon \in (0,\varepsilon_1]$, we have

$$\sup_{\omega \in [0,k\varepsilon^{-\alpha}]} E d_\infty^2 (U_\varepsilon(\omega), V_\varepsilon(\omega)) \leqslant \Delta$$

Proof. For any $\omega \in [0,U] \subset \iota$,

$$\begin{aligned}
& \sup_{\omega\in[0,U]} Ed_\infty^2(U_\varepsilon(\omega),V_\varepsilon(\omega)) \\
= \; & \sup_{\omega\in[0,U]} Ed_\infty^2\Bigg(C_q(\omega)\eta_0+K_q(\omega)\eta_1+\frac{1}{\Gamma(\beta)}\int_0^\omega(\omega-\nu)^{q-1}P_q(\omega-\nu) \\
& \left[\varepsilon\mathscr{B}\int_0^\omega \mathscr{Q}(\nu,U(h(\nu)))d\nu+\sqrt{\varepsilon}\Big\langle \mathscr{H}(\nu,U(\nu))d\mathscr{B}(\nu)\Big\rangle\right], C_q(\omega)\eta_0 \\
& +K_q(\omega)\eta_1+\frac{1}{\Gamma(\beta)}\int_0^\omega(\omega-s)^{q-1}P_q(\omega-s)\left[\varepsilon\mathscr{B}\int_0^\omega \tilde{\mathscr{Q}}(\nu,V(h(\nu)))d\nu+\right. \\
& \left.\left.\sqrt{\varepsilon}\Big\langle \tilde{\mathscr{H}}(\nu,V(\nu))d\mathscr{B}(\nu)\Big\rangle\right]\right) \\
\leqslant \; & 2\varepsilon^2 \sup_{\omega\in[0,u]} Ed_\infty^\omega\Bigg(\int_0^\omega(\omega-s)^{q-1}P_q(\omega-s)B\int_0^\omega \mathscr{Q}(\nu,V(h(\nu)))d\nu, \\
& \int_0^\omega(\omega-\nu)^{q-1}P_q(\omega-\nu)B\int_0^\omega \tilde{\mathscr{Q}}(\nu,V(h(\nu)))ds\Bigg) \\
& +2\varepsilon \sup_{\omega\in[0,U]} Ed_\infty^2\left(\Big\langle\int_0^\omega \mathscr{H}(\nu,U_\varepsilon(\nu))d\mathscr{B}(\nu)\Big\rangle,\Big\langle\int_0^\omega \tilde{\mathscr{H}}(V_\varepsilon(\nu))d\mathscr{B}(\nu)\Big\rangle\right).
\end{aligned}$$

Denote by

$$J_1 = 2\varepsilon^2 \sup_{\omega\in[0,U]} Ed_\infty^2\left(\int_0^\omega \mathscr{Q}(\nu,U(h(\nu)))d\nu, \int_0^\omega \tilde{\mathscr{Q}}(\nu,V(h(\nu)))d\nu\right)$$

and

$$J_2 = 2\varepsilon \sup_{\omega\in[0,U]} Ed_\infty^2\left(\left\langle \int_0^\omega \mathscr{H}(\nu,U_\varepsilon(\nu))d\mathscr{B}(\nu)\right\rangle,\left\langle \int_0^\omega \tilde{\mathscr{H}}(\nu,V_\varepsilon(\nu))d\mathscr{B}(\nu)\right\rangle\right).$$

Then, by applying the assumptions of the metric d_∞, we obtain

$$\begin{aligned} J_1 &\leqslant 4\varepsilon^2 \sup_{\omega\in[0,U]} Ed_\infty^2\left(\int_0^\omega \mathscr{Q}(\nu,U_\varepsilon(h(\nu)))d\nu, \int_0^\omega \mathscr{Q}(\nu,V_\varepsilon(h(\nu)))d\nu\right)+ \\ & \quad 4\varepsilon^2 \sup_{\omega\in[0,U]} Ed_\infty^2\left(\left\langle \int_0^\omega \mathscr{H}(\nu,U_\varepsilon(h(\nu)))d\mathscr{B}(\nu)\right\rangle,\right. \\ & \quad \left.\left\langle \int_0^\omega \tilde{\mathscr{H}}(\nu,V_\varepsilon(h(\nu)))d\mathscr{B}(\nu)\right\rangle\right) := J_{11}+J_{12} \end{aligned}$$

We have used Proposition 2 and the assumption $(\mathscr{A}_2)$ to obtain

$$\begin{aligned} J_{11} &\leqslant 4\varepsilon^2 \sup_{\omega\in[0,U]} \left(\omega \int_0^\omega Ed_\infty^2(\mathscr{H}(\nu,U_\varepsilon(h(\nu))),\mathscr{H}(\nu,V_\varepsilon(h(\nu))))d\nu\right) \\ &\leqslant 4\varepsilon^2 C_2 U \int_0^U Ed_\infty^2(U_\varepsilon(\nu),V_\varepsilon(\nu))d\nu. \end{aligned}$$

We apply Proposition 2 and the assumption $(\mathscr{A}_3)$ to calculate J_{12}.

$$\begin{aligned} J_{12} &\leqslant 4\varepsilon^2 \sup_{\omega\in[0,u]} \left(\omega \int_0^\omega Ed_\infty^2(\mathscr{Q}(\nu,V_\varepsilon(h(\nu))),\tilde{\mathscr{Q}}(\nu,V_\varepsilon(h(\nu))))d\nu\right) \\ &\leqslant 4\varepsilon^2 \sup_{\omega\in[0,u]} \left(\omega^2 \frac{1}{\omega} \int_0^\omega Ed_\infty^2(\mathscr{Q}(\nu,V_\varepsilon(h(\nu))),\tilde{\mathscr{Q}}(\nu,V_\varepsilon(h(\nu))))d\nu\right) \\ &\leqslant 4\varepsilon^2 U^2 \beta_1(U)[1+\sup_{\omega\in[0,U]} Ed_\infty^2(V_\varepsilon(\omega),\hat{0})] \\ &:= 4\varepsilon^2 U^2 \lambda_1. \end{aligned}$$

Therefore,

$$J_1 \leqslant 4\varepsilon^2 C_2 U \int_0^U Ed_\infty^2(U_\varepsilon(\nu),V_\varepsilon(\nu))d\nu + 4\varepsilon^2 U^2 \lambda_1. \tag{5}$$

Using Proposition 1, we have the second term J_2,

$$\begin{aligned}
J_{12} &\leqslant \sup_{\omega\in[0,U]} \int_0^\omega E\|\mathscr{H}(\nu,U_\varepsilon(\nu)) - \tilde{\mathscr{H}}(\nu,V_\varepsilon(\nu))\|^2 d\nu \\
&\leqslant 4\varepsilon \sup_{\omega\in[0,U]} \int_0^\omega E\|\mathscr{H}(\nu,U_\varepsilon(\nu)) - \mathscr{H}(\nu,V_\varepsilon(\nu))\|^2 d\nu \\
&\quad +4\varepsilon \sup_{\omega\in[0,U]} \int_0^\omega E\|\mathscr{H}(\nu,V_\varepsilon(\nu)) - \tilde{\mathscr{H}}(\nu,V_\varepsilon(\nu))\|^2 d\nu \\
&:= J_{21} + J_{22}
\end{aligned}$$

Using the assumption $(\mathscr{A}_2)$, we get

$$J_{21} \leqslant 4\varepsilon C_2 \int_0^U Ed_\infty^2(U_\varepsilon(\nu),V_\varepsilon(\nu))d\nu.$$

Additionally, based on the supposition $(\mathscr{A}_\ni)$, we have

$$\begin{aligned}
J_{12} &\leqslant 4\varepsilon^2 \sup_{\omega\in[0,U]} \left(\omega \frac{1}{\omega} \int_0^\omega E\|(\mathscr{Q}(\nu,V_\varepsilon(h(\nu))),\tilde{\mathscr{H}}(\nu,V_\varepsilon(\nu)))\|^2 d\nu\right) \\
&\leqslant 4\varepsilon U \beta_2(U)[1 + \sup_{\omega\in[0,U]} Ed_\infty^2(V_\varepsilon(\omega),\hat{0})] \\
&:= 4\varepsilon U \lambda_2.
\end{aligned}$$

Therefore,

$$J_2 \leqslant 4\varepsilon C_2 \int_0^U Ed_\infty^2(U_\varepsilon(\nu),V_\varepsilon(\nu))d\nu + 4\varepsilon u \lambda_2. \tag{6}$$

By combining 5 and 6, we get

$$
\begin{aligned}
& \sup_{\omega\in[0,U]} Ed_\infty^2(U_\varepsilon(\omega),V_\varepsilon(\omega)) \\
\leqslant\ & 4\varepsilon u(\lambda_2+\varepsilon u\lambda_1)+4\varepsilon C_2(1+\varepsilon U)\int_0^U Ed_\infty^2(U_\varepsilon(s),V_\varepsilon(v))dv \\
\leqslant\ & 4\varepsilon U(\lambda_2+\varepsilon U\lambda_1)+4\varepsilon C_2(1+\varepsilon u)\int_0^u \sup_{V\in[0,v]} Ed_\infty^2(U_\varepsilon(v),V_\varepsilon(v))dv
\end{aligned}
$$

As a result of applying the Gronwall inequality, we obtain

$$\sup_{\omega\in[0,U]} Ed_\infty^2(U_\varepsilon(\omega),V_\varepsilon(\omega)) \leqslant 4\varepsilon U(\lambda_2+\varepsilon U\lambda_1)e^{4\varepsilon C_2(1+\varepsilon U)}.$$

Choose $\varpi\in(1,2)$ and ℓ is positive s.t $\omega\in[0,\ell\varepsilon^{-\varpi}]\subseteq\iota$,

$$\sup_{\omega\in[0,U]} Ed_\infty^2(U_\varepsilon(\omega),V_\varepsilon(\omega)) \leqslant k\ell\varepsilon^{1-\varpi},$$

where $k=4(\lambda_2+\ell\varepsilon^{1-\alpha}\lambda_1)\exp\{4\varepsilon C_2(1+\ell\varepsilon^{1-\varpi})\}$ is a constant. Consequently, for each given number $\Delta, \exists\varepsilon_1\in(0,\varepsilon_0]$ such that $\varepsilon\in(0,\varepsilon_1]$ and $\omega\in[0,\ell\varepsilon^{-\varpi}]$,

$$\sup_{\omega\in[0,U]} Ed_\infty^2(U_\varepsilon(\omega),V_\varepsilon(\omega)) \leqslant \Delta.$$

■

4. Example

This part contains example to demonstrate our major point of this paper.

Example 1. Consider the FSFEEs listed below:

$$
\begin{aligned}
& {}_0^cD_\omega^\gamma U(\omega)=5sin(\omega)cos(\omega)U(\omega)d\omega+\langle hU(\omega)dB(\omega)\rangle, \\
& U(0)=\eta_0, \\
& U'(0)=\eta_1.
\end{aligned}
\qquad (7)
$$

The standard form of the FSFEEs is

$$ {}_0^cD_\omega^\gamma U^\varepsilon(\omega) = 5sin(\omega)cos(\omega)U^\varepsilon(\omega)d\omega + \langle hU^\varepsilon(\omega)dB(\omega)\rangle. $$

Note that $f(\omega, U^\varepsilon) = 5sin(\omega)cos(\omega)U^\varepsilon(\omega)$ and $g(\omega, U^\varepsilon) = hU^\varepsilon(\omega)$. Hence,

$$ \tilde{f}(U^\varepsilon) = \frac{1}{\pi}\int_0^\pi 5sin(\omega)cos(\omega)U^\varepsilon(\omega)d\omega = 2X^\varepsilon $$

and

$$ \tilde{g}(U^\varepsilon) = \frac{1}{\pi}\int_0^\pi g(\omega, hU^\varepsilon)d\omega = hU^\varepsilon. $$

As a result, the average form of eq (7) is

$$ dV^\varepsilon = 2\varepsilon V^\varepsilon d\omega + \sqrt{\varepsilon}\langle hV^\varepsilon dB(\omega)\rangle. \tag{8} $$

Because the coefficients $f(\omega, U^\varepsilon)$ and $g(\omega, u^\varepsilon)$ fulfil the assumption $(\mathscr{A}_1) - (\mathscr{A}_2)$, FSFEEs 7 has a unique fuzzy solution. Furthermore, the coefficients $\tilde{f}(U^\varepsilon)$ and $\tilde{g}(U^\varepsilon)$ fulfil the assumption (A_3). As a result of Theorem 1, the solutions U^ε and V^ε to Eq. 7 and 8 are equal in mean square concept.

5. Conclusion

We introduce the AP for FSFEEs in this work. We explained that the averaged FSFIEE solution converges to the standard FSFIEE solution in the concept of mean square. In the future, we intend to investigate the topic and compensate the conclusion of the AP for FSFIEEs with and without Hilfer fractional derivative. This field has great potential for numerous research projects that can lead to significant applications and theories. We intend to focus our attention on this area of research.

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