Available Online
International Journal of Advancements in Mathematics
ISSN: 2959-2755 (Print), 2959-2763 (Online)
http://www.scienceimpactpub.com/IJAM

Existence, uniqueness and stability analysis of fractional-order neural networks with multiple delays and variable coefficients: Banach fixed point approach

Kanwal Niazi[1], **Naveed Ahmed**[2], **Aneela Umar**[3†], **Kiran Pasha**[4]

Abstract This study uses the Banach fixed point idea and analysis technique to explore the existence, uniqueness, and stability of solutions for a class of fractional-order neural networks. For fractional-order neural networks with multiple time delays and variable coefficients, a necessary situation is stated to guarantee the uniqueness, existence, and uniform solutions of stability. The outcomes are simple to confirm in practice and, to a certain extent, build upon and extend a number of prior initiatives. An excellent example is given to illustrate how the findings might be applied and relied upon.
Keywords: Fractional evolution equations; mild solution; existence; Continuous dependence; Optimal control

1. Introduction

For a very long time, fractional calculus has attracted the most interest. Fractional calculus is a modern field of study that focuses on the analysis of integrals and deriva-

https://doi.org/10.52223/ijam.2023.312
†Corresponding author: malikannie31@gmail.com
[1] Department of Mathematics and Statistics, The university of Lahore, Sargodha campus
[2] School Education Department, Punjab, Pakistan
[3] The citizen foundations, Khushab, Punjab, Pakistan
[4] Punjab group of Colleges, Quaidabad, Punjab, Pakistan

tives of any order with non-integer value, whether even, rational, or irrational complicated. Fractional calculus has applications in many disciplines, including Chemistry (1), biology (2), Optics (3), statistical mechanics and its applications (4), economics (5), finance (6), electricity (7), mechanics (8), physics (9), and control theory (9). The idea of fractional derivatives is the primary justification for the careful application of the concept of fractional calculus, as opposed to integral-order derivatives, offer an efficient and superior this study presents a methodology to assess the hereditary and memory Integral-order derivatives are used to compare the characteristics of distinct materials and processes. Fractional calculus is playing an increasingly key role in science and engineering fields. The study of fractional calculus has received the majority of interest for a long time. Because fractional-differential equations have been developing steadily in recent years, many academics have started to focus on the theory of fractional orders, and the coupling of fractional orders and neural networks fully emphasises the benefits of fractional orders.

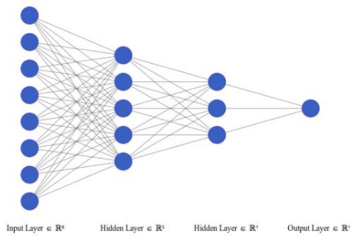

Figure 1
Artificial neural network (ANN)

Figure (1) shows the artificial neural networks. Artificial neural networks (ANN) are usually called neural networks (NNs) or neural nets are computing systems inspired by biological neural network. Many academics have begun to concentrate on the theory of fractional orders as a result of the steady development of fractional-differential equations in recent years, and the coupling of fractional orders with neural networks completely accentuates the advantages of fractional orders. Generally, neurons are arranged into layers. Different layers may perform different transformations on their inputs. Signals move through the layers, perhaps more than once, from the first layer (the input layer) to the last layer (the output layer). Artificial neural network simulate the basic functions of biological neurons: input, passing information to other neurons. A neural network, which refers to a category of artificial intelligence method, facilitates computers to conduct data analysis resembling the cognitive processing of brain of the human. Artificial neural networks are used in a wide range of applications like, image identification, speech recognition, machine translation, and medical diagnosis. Due to the enormous complexity of fractional calculus, the topic of stability has only lately been studied and debated, with relatively few findings that are actually useful. This is true even though it is fundamental and necessary for fractional-order neural networks. It has been shown that neural networks perform especially well in a range of engineering applications (10), (11), (12), (13). The utilization of fractional calculus has garnered increasing significance within the fields of natural and applied sciences and engineering. For an extended period, there has been a significant interest in fractional calculus. Recent years have seen a rise in interest in fractional-order theory as a result of the continuous development of fractional-differential equations, and the coupling of fractional order and neural networks fully emphasizes the benefits of fractional order. Artificial neural networks are now modeled using fractional calculus, and research on biological neurons supports the use of fractional-order formulations in neural network models (14), (16) (17), (18). In their work, The significance of creating and researching fractional-order mathematical models for neural networking has been emphasized by the authors (16).

In (19) Liang and Cao investigate fractional-order Hopfield neural networks' stability and multi-stability in the absence of temporal delays with constant coefficients. An essential prerequisite regarding a group of fractional-order neural networks with continuous delay was established in (21) under the assumption that there are no initial conditions. This requirement ensures the uniqueness, uniform stability, and existence of the equilibrium point. Due to the enormous complexity of fractional calculus, the topic of stability has only lately been studied and debated, with relatively few findings that are actually useful. Even though it is essential and fundamental for fractional-order neural networks. Despite the existence of notable results findings on the Integer-order neural networks' stability with variable coefficients are available, while research on the stability of fractional-order neural networks with variable coefficients is still in its infancy.

Several coefficients could be obtained (21; 22; 23; 24; 25). This research aims to address the aforementioned issues by providing theoretical stability analysis for a class of variable coefficient, multiple time delay, and fractional-order neural networks. In the paper Zou et.al (27) studied the neural network model with numerous time delays and variable coefficients:

$$D^{\alpha}x_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij})) + I_i(t),\ t \in [0,T]$$

where $T < +\infty; D^{\alpha}$ represents the order's Caputo fractional-order derivative $\alpha(0 < \alpha < 1)$; $i = 1,2,....n$ and the letter that represents the number of units in a neural network is "n"; $x_i(t)$ denotes the state of the ith unit at time $t; f_j(x_j(t))$ indicates the mechanisms through which the jth unit at time $t; c_i(t) > 0$ relates to the speed at which, when cut off from the network and external inputs at time, the ith in isolation, the unit will reset its potential to the resting condition $t; a_{ij}(t)$ and $b_{ij}(t)$ represent the intensity of the link between jth unit on the ith unit at time t and $t - \tau_{ij}$, respectively; $I_i(t)$ shows the external inputs at time $t; \tau_{ij}$ corresponds to the transmission delay along the axon of the jth unit, and $0 \leqslant \tau_{ij} \leqslant \tau = max\tau_{ij}|i,j = 1,2,...n$. the initial conditions associated with above equation are of the form: $x_i(t) = \phi_i(t), t \in [-\tau,0]$, $i = 1,2,...n$, where $\phi_i(t) \in C([-\tau,0],R)$, and the norm of an component in $C([-\tau,0],R^n)$ is $||\phi|| = \sum_{i=1}^{n} sup_{t\in[-\tau,0]} e^{-Nt}|\phi_i(t)|$ motivated by this system we will try to work for a family of fractional order-neural networks on stability analysis.

This is how the paper is set up. Section 2 provides a number of fundamental ideas and lemmas for fractional calculus. The main results are drawn from a description of fractional-order neural networks with varying coefficients and various time delays in Section 3. In Section 4,5 a case study is provided to illustrate the conclusions of this study. Section 6 forms some conclusions.

2. Preliminaries

The area is described $\Omega = (C([0,T],R^n),(||.||))$ a Banach space, in which $C([0,T],R^n)$ consists of all uninterrupted columns n-vector function classes. For $\xi \in C([0,T],R^n)$, the norm is defined by $||\xi|| = \Sigma_{i-1}^{n} sup_u\{e^{-Nu}|\xi_{i(u)}|\}$. Besides, for a matrix $A = (u_{ij}(u))_{n\times n}$, we define the norm

$$||A|| = \sum_{i-1}^{n} u_i = \sum_{i-1}^{n} \sup_{u,\forall j} |u_{ij}(u)|.$$

Definition 2.1. (25) An integral fuction of fractional order $f(u)$ of order $\alpha \in R^+$ is defined by:

$$I_{u_0}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_{u_0}^{u} \frac{f(\tau)}{(u-\tau)^{1-\alpha}} d\tau, \tag{1}$$

Where $\Gamma(\cdot)$ is the gamma function defined as:

$$\Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u} du. \tag{2}$$

Definition 2.2. (26) The fractional derivative of Caputo D^{α} of α order of a function $f(u)$ is given as:

$$D_{u_0}^{\alpha} f(u) = \frac{1}{\Gamma(n-\alpha)} \int_{u_0}^{u} \frac{f^{(n)}(\tau)}{(u-\tau)^{1+\alpha-n}} d\tau, \tag{3}$$

Lemma 2.1. *(25) If the fractional derivative of the caputo $D_{u_0}^{\alpha} f(u)(n-1) \leqslant \alpha < n$ is integrable, then:*

$$I_{u_0}^{\alpha} D_{u_0}^{\alpha} f(u) = f(u) - \sum_{i=0}^{n-1} \frac{f^(i)(u_0)}{i!} (u-u_0)^i. \tag{4}$$

one can obtain: In particular, the following is available for $0 < \alpha < 1$:

$$I_{u_0}^{\alpha} D_{u_0}^{\alpha} f(u) = f(u) - f(u_0). \tag{5}$$

Lemma 2.2. *(Gronwall inequality).(28) If $x(u), f(u), g(u) \geqslant 0$ is a continuous function* *on*$(0,T], T < \infty$ *and satisfies the following inequality*

$$x(u) \leqslant f(u) + \int_0^u g(\mu)x(\mu)d\mu, t \in [0,T) \tag{6}$$

In special cases, if f(u)is a non increasing function, we can get

$$x(u) \leqslant f(u)\exp\int_o^u g(v)dv, t \in [0,T)$$

3. Research method

A neural network model with varying coefficients and several time delays is shown below:

$$D^\alpha x_i(u) = Bx_i(u) + f(u,x_i(u)) + c\int_{-\tau_1}^0 x_i(u+\theta)d\theta + I_i(u) + \Delta I_i(u) \tag{7}$$

$T < +\infty$; where D^α represents the caputo order of fractional derivative. B is the constant matrix, f denotes the activation function. $x_i(u)$ denotes the ith unit's state at time u. $I_i(u)$ denotes the external inputs at time u. $\Delta I_i(u)$ is the input disturbance in the controller. The ith unit will unplug from the network and all external inputs, resetting its potential to its idle state and initial condition for equation (7) is:

$$x_i(u) = \phi_i(u), \ u \in [-\tau_1, 0], \ i = 1,2,3,...,n. \tag{8}$$

where $\phi_i(u) \in C([-\tau_1, 0], R)$, and $R^n)$ is $\|\phi\| = \sum_{i=1}^n \sup_{u\in[-\tau,0]} \{e^{-Nu}|\phi_i(u)|\}$.
we make the following assumptions throughout this paper to reach our results.
Assumption 1: $x_i(u)$ and $I_i(u)$ are continuous on $[0,T]$.
Assumption 2: The activating functions f have positive constants and are Lipschitz continuous c_1 such that:

$$|f(u,x_i(u)) - f(u,y_i(u))| \leqslant c_1|x_i(u) - y_i(u)| \ for \ all \ x_i, \ y_i \in R, \ i = 1,2,....n, \tag{9}$$

4. Existence and Uniqueness

Theorem 4.1. *The system (7) has a unique solution if assumptions (1) and (2) hold and the system satisfies initial conditions (8)*

$$x(u) = (x_1(u), x_2(u), \ldots, x_n(u))^T \in C([0,T], R^n).$$

Proof. We can write solution of equation (7) by using properties of fractional calculus in the form of appropriate volterra integral equation:

$$x_i(u) = \Phi_i(0) + I^\alpha D^\alpha x_i(u)$$

$$= \Phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \left[Bx_i(u) + f(u, x_i(u)) + C \int_{-\tau_1}^0 x_i(u+\theta) d\theta + I_i(u) \right] dv \tag{10}$$

We transform the problem (7) into a fixed problem and consider a mapping defined by:

$$F : R^n \to R^n \ \ where\ F_x = (F_1 x_1, F_2 x_2 \ldots . F_n x_n)^T$$

$$\begin{aligned} F_i x_i(u) \quad &= \quad \Phi_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (u-v)^{\alpha-1} \Big[Bx_i(u) + f(u, x_i(v)) \\ &\quad + C \int_{-\tau_1}^0 x_i(u+\theta) d\theta + I_i(u) \Big] dv \end{aligned} \tag{11}$$

for any two different functions

$$\begin{aligned} x(u) &= (x_1(u), x_2(u), \ldots x_n(u))^T and \\ y(u) &= (y_1(u), y_2(u), \ldots y_n(u))^T \end{aligned}$$

we have,

$$\begin{aligned} |f_i x_i(u) - f_i y_i(u)| &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \Big[B|x_i(u) - y_i(u)| + f(u,x_i(u)) \\ &\quad -f(u,y_i(u)) + C \int_{-\tau_1}^0 (x_i(u+\theta)) - y_i(u+\theta) d\theta \Big] dv \end{aligned}$$

$$\begin{aligned} \max_{0 \leqslant v \leqslant u} |x_v - y_v| &= \max_{0 \leqslant v \leqslant u} |x(v+\theta) - y(v+\theta)| \\ &= \max_{0 \leqslant v+\theta \leqslant u+\theta} |x(v+\theta) - y(v+\theta)| \\ &\leqslant \max_{-\tau \leqslant \tilde{v} \leqslant 1} |x(\tilde{v}) - y(\tilde{v})| \\ &= ||x-y|| \end{aligned}$$

where $v+\theta = \tilde{v}, f(v) = c_1$ and $-\tau \leqslant \theta < 0$
we can use this term in the onward equations, now we can write the above equation as:

$$\begin{aligned} &\leqslant \frac{1}{\Gamma(\alpha)} \sup_u B \Big[|x_i(u) - y_i(u)| \int_0^u (u-v)^{\alpha-1} dv + \frac{1}{\Gamma(\alpha)} \sup_u f(v) \\ &\quad |x_i(u) - y_i(u)| \int_0^u (u-v)^{\alpha-1} dv + \frac{1}{\Gamma(\alpha)} C \sup_u \int_0^u (u-v)^{\alpha-1} \\ &\quad \int_{-\tau_1}^0 (x_i(u+\theta) - y_i(u+\theta)) d\theta \Big] dv \\ &= \frac{1}{\Gamma(\alpha)} \sup_u B |x_i(u) - y_i(u)| \int_0^u (u-v)^{\alpha-1} dv + \frac{1}{\Gamma(\alpha)} c_1 |x_i(u) - y_i(u)| \\ &\quad \int_0^u (u-v)^{\alpha-1} dv + \frac{1}{\Gamma(\alpha)} C \int_0^u (u-v)^{\alpha-1} \int_{-\tau_1}^0 |x(\tilde{v}) - y(\tilde{v})| d\theta dv \\ &= \frac{B}{\Gamma(\alpha)} |x_i(u) - y_i(u)| \int_0^u (u-v)^{\alpha-1} dv + \frac{u^\alpha}{\Gamma(\alpha)} c_1 |x_i(u) - y_i(u)| \int_0^u (u-v)^{\alpha-1} dv \\ &\quad - \frac{c\tau_1}{\Gamma(\alpha)} |x-y| \int_0^u (u-v)^{\alpha-1} dv e^{-Nu} |F_i x_i(u) - F_i x_i(u)| \end{aligned}$$

$$
\begin{aligned}
\leqslant\ & \frac{B}{\Gamma(\alpha)}|x_i(v)-y_i(v)|\int_0^u (u-v)^{\alpha-1} + \frac{u^\alpha}{\Gamma(\alpha)}c_1|x_i(v)-y_i(v)|\int_0^u (u-v)^{\alpha-1}dv \\
& -\frac{c\tau_1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}dv
\end{aligned}
$$

which implies that

$$
\begin{aligned}
=\ & \frac{B}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-Nv}|x_i(v)-y_i(v)|dv \\
& +\frac{1}{\Gamma(\alpha)}c_1u^\alpha\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-}Nv|x_i(v)-y_i(v)|dv \\
& -\frac{1}{\Gamma(\alpha)}c\tau_1\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-Nv}|x_i(v)-y_i(v)|dv \\
\leqslant\ & B\sup_u e^{-Nu}|x_i(v)-y_i(v)|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv \\
& +c_1u^\alpha\sup_u e^{-Nu}|x_i(u)-y_i(u)|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv \\
& -C\tau_1\sup_u e^{-Nu}|x_i(u)-y_i(u)|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv \qquad (12)
\end{aligned}
$$

Of course, there are

$$
\begin{aligned}
||Fx(u)-Fy(u)|| &= \sup_u e^{-Nu}|F_ix_i(u)-F_iy_i(u)| \\
&\leqslant \frac{B}{N^\alpha}\sup_u|x_i(u)-y_i(u)| + \frac{c_1u^\alpha}{N^\alpha}||x(u)-y(u)|| - \frac{c\tau_1}{N^\alpha}||x(u)-y(u)|| \\
&\leqslant (\frac{B}{N^\alpha}+\frac{c_1u^\alpha}{N^\alpha}-\frac{c\tau_1}{N^\alpha})||x(u)-y(u)||
\end{aligned}
$$

Now select N large enough so that

$$
(B+(||c_1u^\alpha||-||c\tau_1||)<N^\alpha \qquad (13)
$$

Then we have,
$||Fx(u)-Fy(u)||<||x(u)-y(u)||$ ■

5. Stability

Definition: If for any reason the solution to system (7) is stable, $\varepsilon > 0$, $u_0 \geqslant 0$, there is a comparable value available. $\delta(\varepsilon, u_0) > 0$ such that $||y(u, u_0, \phi) - x(u, u_0, \phi)|| < \varepsilon$ for $u \geqslant u_0$ as once the first criteria are met $||\varphi(u) - \phi(u)|| < \delta(\varepsilon, u_0)$. If the preceding is true, the solution of equation (7) is known to be stable as uniformly δ can be selected separately from u_0 : $\delta(\varepsilon, u_0) \equiv \delta(\varepsilon)$.

Theorem 5.1. *The system solution provided by equation (7) that satisfies the starting condition (8) is uniformly stable assuming assumptions 1 and 2 are satisfied.*

Proof. Let $x(u) = ((x_1(u), x_2(u), \ldots, x_n(u))^T and y(u) = (y_1(u), y_2(u), \ldots, y_n(u))^T$ be two solutions to equation (7), each with a unique beginning condition $x_i(u) = \phi_i(u), y_i(u) = \varphi_i(u), i = 1, 2, \ldots n$. Then for $t \in [0, T]$,We have:

$$
\begin{aligned}
D^{\alpha}(y_i(u) - x_i(u)) &= B(y_i(u) - x_i(u)) + f(u, y_i(u) - f(u, x_i(u)) + c\int_{-\tau_1}^{0} y_i(u+\theta) \\
&\quad -x_i(u+\theta)d\theta
\end{aligned} \tag{14}
$$

Its form is as follows and it shares the same structure as the nonlinear Volterra integral equation:

$$
\begin{aligned}
y_i(u) - x_i(u) &= \varphi_i(0) - \phi_i(0) + I^{\alpha}\Big[B(y_i(u) - x_i(u)) + f(u, y_i(u) \\
&\quad -f(u, x_i(u)) + c\int_{-\tau_1}^{0} y_i(u+\theta) - x_i(u+\theta)d\theta\Big]ds \\
&= \varphi_i(0) - \phi i(0) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}[B(y_i(u) - x_i(u)) + f(u, y_i(u) \\
&\quad -f(u, x_i(u)) + c\int_{-\tau_1}^{0} y_i(u+\theta) - x_i(u+\theta)d\theta]dv \\
&\leqslant e^{-Nu}|\varphi_i(0) - \phi_i(0)| + \frac{1}{\Gamma(\alpha)}e^{-Nu}\int_0^u (u-v)^{\alpha-1}\Big[B|y_i(v) - x_i(v)| \\
&\quad +c_1||y_i(v) - x_i(v)|| - |c|\int_{-\tau_1}^{0} y_i(v+\theta) - x_i(v+\theta)dv\Big]
\end{aligned}
$$

$$\leqslant \quad e^{-Nu}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}e^{-Nu}\sup_u B\int_0^u (u-v)^{\alpha-1}|(y_i(v)-x_i(v)|dv$$
$$+\frac{1}{\Gamma(\alpha)}e^{-Nu}sup_u|c_1u^\alpha|\int_0^u (u-v)^{\alpha-1}|y_i(v)-x_i(v)|dv$$
$$+\frac{1}{\Gamma(\alpha)}\sup_u|c\tau_1|\int_{-\tau_1}^0 (u-v)^{\alpha-1}|y_i(v+\theta)-x_i(v+\theta)|dv$$

$$e^{-Nu}|y_i(u)-x_i(u)|$$
$$\leqslant \quad e^{-Nu}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}e^{-Nu}B\int_0^u (u-v)^{\alpha-1}$$
$$+\frac{1}{\Gamma(\alpha)}e^{-Nu}C_1u^\alpha\int_0^u (u-v)^{\alpha-1}|y_i(v)-x_i(v)|dv$$
$$-\frac{1}{\Gamma(\alpha)}e^{-Nu}C\tau_1\int_0^u (u-v)^{\alpha-1}|y_i(v+\theta)-x_i(v+\theta)|dv$$
$$\leqslant \quad e^{-Nu}|\varphi_i(0)-\phi_i(0)|+\frac{1}{\Gamma(\alpha)}B\int_0^u (u-v)^{\alpha-1}e^-N(u-v)e^{-Nv}|y_i(v)-x_i(v)|dv$$
$$+\frac{1}{\Gamma(\alpha)}C_1u^\alpha\int_0^u (u-v)^{\alpha-1}$$
$$e^{-N(u-v)}e^{-Nv}|y_i(v)-x_i(v)|dv-\frac{1}{\Gamma(\alpha)}C\tau_1\int_0^u (u-v)^{\alpha-1}e^{-N(u-v+\theta)}e^{-N(v-\theta)}$$
$$|y_i(v+\theta)-x_i(v+\theta)|dv \tag{15}$$

From (15), we get:

$$\leqslant \quad e^{-Nu}|\varphi_i(0)-\phi_i(0)|+\frac{B}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^-N(u-v)e^{-Nv}|y_i(v)-x_i(v)|dv$$
$$+\frac{C_1u^\alpha}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-Nv}|y_i(v)-x_i(v)|dv$$
$$+\frac{C\tau_1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v+\theta)}e^{-N(v-\theta)}|y_i(v+\theta)-x_i(v+\theta)|dv$$

$$
\begin{aligned}
\leqslant \quad & e^{-Nu}|\varphi_i(0)-\phi_i(0)| + \frac{B}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-Nv}|y_i(v)-x_i(v)|dv \\
& +\frac{C_1 t^{\alpha}}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}e^{-Nv}|y_i(v)-x_i(v)|dv \\
& -\frac{C\tau_1}{\Gamma(\alpha)}\int_0^{\theta} (u-v)^{\alpha-1}e^{-N(u-v+\theta)}e^{-N(v-\theta)}|y_i(\tilde{s})-x_i(\tilde{s})|dv \\
& -\frac{C\tau_1}{\Gamma(\alpha)}\int_{\theta}^{u} (u-v)^{\alpha-1}e^{-N(u-v+\theta)}e^{-N(v-\theta)}|y_i-x_i|dv \\
\leqslant \quad & \sup_{u\in[-\tau_1,0]} e^{-Nu}|\varphi_i(0)-\phi_i(0)| + B\sup_u e^{-Nu}|y_i(u)-x_i(u)|\frac{1}{\Gamma(\alpha)} \\
& \int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv + c_1 u^{\alpha}\sup_u e^{-Nu}|y_i(u)-x_i(u)| \\
& \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv \\
& -c\tau_1 \sup_{u\in[0,\theta]} e^{N(u-\theta)}|y_i(u+\theta)-x_i(u+\theta)|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv] \\
& -c\tau_1[\sup_u \int_0^{\theta} e^{N(u-\theta)}|y_i-x_i|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}e^{-N(u-v)}dv] \\
\leqslant \quad & \sup_{u\in[-\tau_1,0]} e^{-Nu}|\varphi_i(0)-\phi_i(0)| + \frac{B}{N^{\alpha}}\sup_u e^{-Nu}|y_i(u)-x_i(u)| \\
& +c_1 u^{\alpha}||y(u)-x(u)|| - \frac{c\tau_1}{N^{\alpha}}\sup_u \in [-\tau_1,0]e^{-Nu}|\varphi_i(u)-\phi_i(u)| \\
& -\frac{c\tau_1}{N^{\alpha}}||y_i-x_i|| \\
\leqslant \quad & \sup_{u\in[-\tau_1,0]} e^{-Nu}|\varphi_i(0)-\phi_i(0)| + \frac{B}{N^{\alpha}}\sup_u e^{-Nu}|y_i(u)-x_i(u)| + c_1 u^{\alpha}||y(u)-x(u)|| \\
& -\frac{c\tau_1}{N^{\alpha}}||y(u)-x(u)|| - \frac{c\tau_1}{N^{\alpha}}||\varphi(u)-\phi(u)||
\end{aligned}
$$

$$\leqslant \sup_{u\in[-\tau_1,0]} e^{-Nu}|\varphi_i(0)-\phi_i(0)| + \frac{B}{N^\alpha}\sup_u e^{-Nu}|y_i(u)-x_i(u)|$$
$$+\frac{c_1u^\alpha - c\tau_1}{N^\alpha}||y(u)-x(u)|| - \frac{c\tau_1}{N^\alpha}||\varphi_i(u)-\phi_i(u)||$$

then we have

$$\begin{aligned}
||y(u)-x(u)|| &= \sum_{i=1}^{n}\sup_u e^{-Nu}|y_i(u)-x_i(u)| \\
&\leqslant \sum_{i=1}^{n}\sup_{u\in[-\tau_1,0]} e^{-Nu}||\varphi(u)-\phi(u)|| + \frac{B}{N^\alpha}\sum_{i=1}^{n}\sup_u|y(u)-x(u)| \\
&\quad +||y(u)-x(u)||\sum_{i=1}^{n}\frac{c_1}{u^\alpha - c\tau_1}N^\alpha - \sum_{i=1}^{n}\frac{c\tau_1}{N^\alpha}||\varphi(u)-\phi(u)|| \qquad (16) \\
&= 1-\frac{||c||\tau_1}{N^\alpha} + B||\varphi(u)-\phi(u)|| + (|\varphi_i(u)-\phi_i(u)|) \\
&\quad +\frac{B}{N^\alpha}\sum_{i=1}^{n}\sup_u|y_i(u)-x_i(u)|(||c_1||u^\alpha - ||c||\tau_1N^\alpha)||y(u)-x(u)||
\end{aligned}$$

It is evident from (16) that

$$||y(u)-x(u)|| \leqslant \frac{1-\frac{||c||\tau_1}{N^\alpha}}{1-\frac{B+||c_1||u^\alpha-||c||\tau_1}{N^\alpha}}||\varphi(u)-\phi(u)||$$

$$||x(u)-y(u)|| < \varepsilon,\ ||\varphi(u)-\phi(u)|| < \delta$$

■

Example 5.2. To contrast the key findings examined in this work with those suggested in reference [(18)], an exemplary case is provided. Think about a group of fractionally ordered delayed neural networks that are defined by the differential equation below.

$$\begin{cases} D^{\alpha}x_1(u) & = 2x_1(u) + 0.84f_1(x_1(u)) - 0.25f_29(x_2(u)) - 0.25f_1(x_1(u+\theta) \\ & +0.1f_2(x_2(u+\theta) - 1.9 + 20cos(0.2) \\ D^{\alpha}x_2(u) & = x_2(u) - 0.15f_1(x_1(u)) + 0.5f_2(x_2(u)) - 0.25f_1(x_1(u+\theta)) \\ & -0.8f_2(x_2(u+\theta)) + 1.6 + 20cos(0.2) \end{cases} \tag{17}$$

where the fractional order α is choosen as $\alpha = 0.5$, the functions are explained by $f_1(x) = f_2(x) = |x+0.8| - |x-0.6|$ and with a time delay $\tau = 0.03$.
Obviously, in system (17) $||c|| = 1.75$, $||B|| = 0.65$, in the aforementioned conditions, we have$||c_1||u^{\alpha} - ||C||\tau_1 > min1 - C, C_1$, However,
system 17 has a special uniform stable solution that in the both theorems solved in this paper.
System (17) actually has a single fixed point that fulfils:

$$\begin{cases} 2x_1^* + 0.59f_1(x_1^*) - 0.15f_2(x_2^*) - 18.09 = 0 \\ x_2^* - 0.4F_1(x_1^*) - 0.3f_2(x_2^*) + 21.59 = 0 \end{cases} \tag{18}$$

Matlab allows us to calculate what the fixed point is $x^* = (0.067, 0.358)^T$.
In the time domain, Figure 2 shows that System (17)'s solution converges to the fixed point. $x*$.

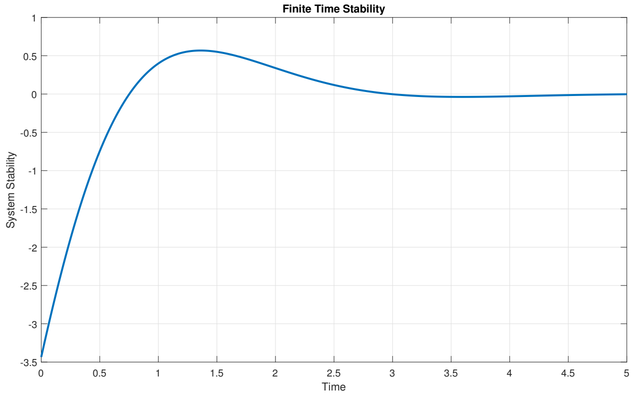

Figure 2
The dynamic behaviour of system (17)

6. Conclusion

This study deals with the uniform stability problem for a class of fractional-order neural networks with variable coefficients and several time delays. It establishes a set of requirements for the availability, uniqueness, and consistent stability of solutions for this kind of neural networks. Lastly, a concrete example is provided for highlighting the value of our findings.

References

[1] Miškinis P. Modelling linear reactions in inhomogeneous catalytic systems. Journal of Mathematical Chemistry. 2013; 51(3): 914-926

[2] García-González MA, Fernández-Chimeno M, Capdevila L, et al. An application of fractional differintegration to heart rate variability time series. Computer methods and programs in biomedicine. 2013; 111(1): 33-40.

[3] Chen D, Sheng H, Chen YQ, Xue D. Fractional-order variational optical flow model for motion estimation. Philosophical Transactions of the Royal

Society A: Mathematical, Physical and Engineering Sciences. 2012; DOI 10.01098/rsta.2012.0148.

[4] Gu H, Liang JR, Zhang YX. Time-changed geometric fractional Brownian motion and option pricing with transaction costs. Physica A: Statistical Mechanics and its Applications. 2012; 391(15): 3971- 3977.

[5] Rostek S, Schöbel R. A note on the use of fractional Brownian motion for financial modeling. Economic Modelling. 2013; 30: 30-35.

[6] Wang FQ, Ma XK. Transfer function modeling and analysis of the open-loop Buck converter using the fractional calculus. Chinese Physics B. 2013; 22(3): 030506.

[7] Xu Z, Chen W. A fractional-order model on new experiments of linear viscoelastic creep of Hami Melon. Computers & Mathematics with Applications. 2013; DOI 10.1016/j.camwa.2013.01.033.

[8] Hilfer R. Applications of fractional calculus in physics. Singapore: World Scientific Singapore. 2000.

[9] Monje CA, Chen YQ, Vinagre BM, et al. Fractional-order Systems and Controls. Berlin: Springer. 2010.

[10] Harikrishna D, Srikanth NV. Dynamic stability enhancemant of power system using neural network controlled static-compensator. TELKOMNIKA Indonesia Journal of Electrical Engineering. 2012; 10(1): 9-16.

[11] Lin N, Li JS. Adaptive neural networks generalized predictive control for unknown nonlinear system. TELKOMNIKA Indonesia Journal of Electrical Engineering. 2013; 11(7): 3611-3617.

[12] Li YD, Zhu L, Sun M. Adaptive RBFNN formation controll of multimobile robots with actuator dynamics. TELKOMNIKA Indonesia Journal of Electrical Engineering. 2013; 11(4): 1797-1806.

[13] Anastasio TJ. The fractional-order dynamics of brainstem vestibulo-oculomotor neurons. Biological Cybernetics. 1994; 72(1): 69-79.

[14] Lundstrom BN, Higgs MH, Spain WJ, et al. Fractional differentiation by neocortical pyramidal neurons. Nature neuroscience. 2008; 11(11): 1335-1342.

[15] Caponetto R, Dongola G, Pappalardo F. Fractional-order simulation tool for the brainstem vestibulo ocular reflex (VOR). Signal, Image and Video Processing. 2012; 6(3): 429-436.

[16] Anastassiou GA. Fractional neural network approximation. Computers & Mathematics with Applications. 2012; 64(4): 1655-1676.

[17] Kaslik E, Sivasundaram S. Nonlinear dynamics and chaos in fractional-order neural networks. Neural Networks. 2012; 32: 245-256.

[18] Chen LP, Chai Y, Wu RC, et al. Dynamic analysis of a class of fractional-order neural networks with delay. Neurocomputing. 2013; 111: 190-194.

[19] Liang J, Cao J. Boundedness and stability for recurrent neural networks with variable coefficients and time-varying delays. Physics Letters A. 2003; 318(1): 53-64.

[20] Jiang H, Teng Z. Global eponential stability of cellular neural networks with time-varying coefficients and delays. Neural Networks. 2004; 17(10): 1415-1425.

[21] Jiang H, Zhang L, Teng Z. Existence and global exponential stability of almost periodic solution for cellular Neural Networks with variable coefficients and time-varying delays. IEEE Transactions on Neural Networks. 2005; 16(6): 1340-1351.

[22] Liu B. Exponential convergence for a class of delayed cellular neural networks with time-varying coefficients. Physics Letters A. 2008; 372(4): 424-428.

[23] Yi X, Shao J, Xiao B. New convergence behavior of high-order Hopfield neural networks with time varying coefficients. Journal of Computational and Applied Mathematics. 2008; 219(1): 216-222.

[24] Tan M, Tan Y. Global exponential stability of periodic solution of neural network with variable coefficients and time-varying delays. Applied Mathematical Modelling. 2009; 33(1): 373-385.

[25] Podlubny I. Fractional differential equations. San Diego: Academic press. 1999.

[26] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam,

[27] Zou, T., Qu, J., Chen, L., Chai, Y., & Yang, Z. (2014). Stability analysis of a class of fractional-order neural networks. TELKOMNIKA Indonesian Journal of Electrical Engineering, 12(2), 1086-1093.

[28] Rus I.A., Gronwall lemmas: ten open problems, Sci. Math. Jpn., 2009, 70(2), 221–228